



PHD

Stochastic processes in random environment

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Stochastic Processes in Random Environment

submitted by

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for the degree of Doctor of Philosophy

of the

University of Bath

Department of Mathematical Sciences

September 2009

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Summary

We are interested in two probabilistic models of a process interacting with a random environment. Firstly, we consider the model of directed polymers in random environment. In this case, a polymer, represented as the path of a simple random walk on a lattice, interacts with an environment given by a collection of time-dependent random variables associated to the vertices. Under certain conditions, the system undergoes a phase transition from an entropy-dominated regime at high temperatures, to a localised regime at low temperatures. Our main result shows that at high temperatures, even though a central limit theorem holds, we can identify a set of paths constituting a vanishing fraction of all paths that supports the free energy. We compare the situation to a mean-field model defined on a regular tree, where we can also describe the situation at the critical temperature.

Secondly, we consider the parabolic Anderson model, which is the Cauchy problem for the heat equation with a random potential. Our setting is continuous in time and discrete in space, and we focus on time-constant, independent and identically distributed potentials with polynomial tails at infinity. We are concerned with the long-term temporal dynamics of this system. Our main result is that the periods, in which the profile of the solutions remains nearly constant, are increasing linearly over time, a phenomenon known as ageing. We describe this phenomenon in the weak sense, by looking at the asymptotic probability of a change in a given time window, and in the strong sense, by identifying the almost sure upper envelope for the process of the time remaining until the next change of profile. We also prove functional scaling limit theorems for profile and growth rate of the solution of the parabolic Anderson model.

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Chapter 1

Introduction

Stochastic processes in random environment inspired a considerable amount of research over the last 50 years. The reason is that introducing a random environment can completely change the nature of a model. The first classical example is the discovery of Anderson [And58] that in a quantum mechanical setting, electrons moving on a lattice structure become trapped when subjected to a random potential. This sharply contrasts their behaviour in an ideal crystal, which is always conductive, see e.g. [Hun08]. The classical probabilistic example is the random walk in random environment, where the transition probabilities are random. Under certain conditions, the random walk becomes trapped and moves significantly slower than in the classical setting, see e.g. [Sin83].

The general set-up includes two source of randomness: first of all a motion (for example a random walk) and secondly a random environment which influences the behaviour of the random walk. One way of studying such a model is by first generating the environment and then investigating the behaviour of the motion given the realisation of the environment. The central question in this setting is: Can the motion “feel” the effect of the random environment? In other words, does the additional randomness in the environment produce effects that cannot be observed otherwise?

In the first part of this thesis, we will concentrate on the model of *directed polymers in random environment*. We aim to model a polymer chain that interacts with randomly distributed impurities in the environment. Once the environment has been generated, we associate an energy to each possible polymer configuration and then pick a polymer randomly while giving a preference to those polymers with a low energy. The model is subjected to a phase transition controlled by a temperature parameter. At high temperatures, we find ourselves in a scenario where the random environment does not have a strong effect. But once the temperature has been decreased below a critical value, we move to a phase where the environment forces the polymers to localize. We attempt to improve our understanding of both phases. In particular, we will see that even in the situation when the environment is weak, not all possible paths are equally important.

This part of the thesis is joint work with *Peter Mörters* and the results about the mean field model on trees have been published in

[MO08] P. Mörters and M. Ortgiere. Minimal supporting subtrees for the free energy of polymers on disordered trees. *J. Math. Phys.* **49**, 2008.

The second part of the thesis deals with the *parabolic Anderson model*, i.e. the heat equation with a random potential. Starting with all the mass at the origin, we observe the heat flow as it interacts with sources of randomly distributed strength. Even though, once the environment is fixed, the flow is not stochastic, it can be interpreted as an average over a stochastic process. This interpretation opens up a new perspective, as the problem can now be considered as a competition between the smoothing effect of the stochastic process versus the irregularity of the environment of random sources. In this case, the random environment changes the model considerably and produces effects that cannot be observed otherwise. In particular, the parabolic Anderson model is one of the most basic models to exhibit the effect of intermittency. Loosely speaking, the main contributions to the total mass of the solution come from small islands that are spatially well-separated. Our main focus, however, is on the effect of ageing. Heuristically this means that the length of the time that the systems stays in the same state increases linearly with time. This part of the thesis is based on joint work with *Peter Mörters* and *Nadia Sidorova* and has resulted in the preprint

[MOS09] P. Mörters, M. Ortgiere and N. Sidorova. Ageing in the parabolic Anderson model. *Preprint*, 2009.

The remainder of the thesis is structured as follows. In this chapter, we will first introduce the random polymer model in Section 1.1 and present our results. Then, in Section 1.2, we will discuss the effect of ageing for the parabolic Anderson model. Chapter 2 contains the proofs for the polymer model, while the final Chapter 3 presents the proofs for the parabolic Anderson model.

1.1 Directed polymers in random environment

The model of directed polymers in random environment was first introduced in the physics literature in [HH85] as a model of an interface in an Ising model with random interactions. Later on, the first rigorous results appeared in [IS88, Bol89], where the model was formulated in the form that we will present below. We will distinguish two models:

- The *lattice model*: The original model of directed polymers in random environment.
- The *tree model*: A mean field model defined on a regular tree introduced by [DS88].

In Section 1.1.1, we will first introduce the original model and review the most important results. Then we will discuss a mean-field model defined on regular trees in Section 1.1.2 and present our main results in the simpler setting. Finally in Section 1.1.3, we will investigate what carries over if we transfer our results to the original model. In particular, we will be able to see when the original model exhibits similar characteristics to the mean field model, but also when the geometry of the underlying space makes a significant difference.

1.1.1 The lattice model

We want to model a hydrophilic polymer chain wafting in water which contains randomly distributed impurities. We represent the polymer chain as the (directed) path of a simple random walk on \mathbb{Z}^d , see Figure 1-1. In other words, a polymer is of the form $(j, \omega_j)_{j=1}^n$, where $(\omega_j)_{j=1}^n$ is the path of a simple random walk in \mathbb{Z}^d , so that $|\omega_j - \omega_{j-1}| = 1$ for all $j = 1, \dots, n$. The random environment is given by a collection of independent, identically distributed random variables $(V(j, x) : j \in \mathbb{N}, x \in \mathbb{Z}^d)$. In the example of the polymer chain in water, each $V(j, x)$ can represent either a site that contains a water molecule, in which case $V = +1$ or an impurity, when $V = -1$. Given this random environment we associate to each polymer (j, ω_j) an energy

$$-\sum_{j=1}^n V(j, \omega_j),$$

The hydrophilic polymer tries to position itself in such a way that it covers many sites where the environment is particularly large, in other words it minimizes its energy. In order to formalize this idea, we define a (finite-volume) *polymer or Gibbs measure* by

$$\mu_n^{(\beta)}((j, \omega_j)_{j=1}^n) = \frac{1}{Z_n(\beta)} e^{\beta \sum_{j=1}^n V(j, \omega_j)},$$

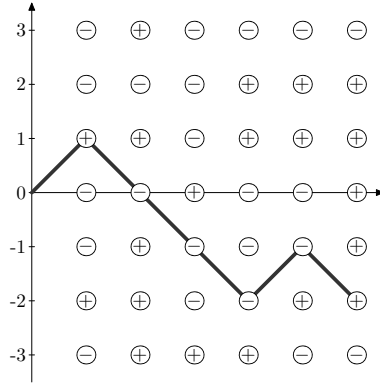


Figure 1-1: A directed polymer in \mathbb{Z}^{1+1} , i.e. $d = 1$, here the environment takes values $+1$ and -1 . The polymer tries to position itself in such a way that it collects mostly $+1$'s and avoids -1 's.

where $\beta > 0$ is the inverse temperature and $Z_n(\beta)$ is a normalizing constant, known as the *partition function*, given by

$$Z_n(\beta) = \sum_{\text{polymers } (j, \omega_j)_{j=1}^n} e^{\beta \sum_{j=1}^n V(j, \omega_j)}.$$

Note, that if $\beta = 0$, then $\mu_n^{(\beta)}$ is the uniform measure on all polymers of length n . As the parameter β increases (the temperature decreases), the effect of the environment becomes stronger. A natural question is at what point is the disorder strong enough to provide a picture that is qualitatively different from the uniform case ($\beta = 0$).

Throughout, we will assume that V is defined on the same probability space as the collection $\{V(n, x) : n \in \mathbb{N}, x \in \mathbb{Z}^d\}$ and has the same distribution as a single member of the collection. We denote the corresponding probability measure by \mathbb{P} and the expectation by \mathbb{E} . Finally, we also assume that V has all exponential moments, i.e.

$$\mathbb{E}[\exp(\beta V)] < \infty \quad \forall \beta \in \mathbb{R}.$$

The first result on this topic, see [IS88, Bol89], shows that in dimensions $d > 2$, and for positive, but small β , we find that under the measure μ_n^β polymers behave diffusively. In other words, in high dimensions and for β small, the polymer behaves as if the random environment was switched off. Bolthausen [Bol89] noticed that the normalized partition function

$$M_n^{(\beta)} = \frac{1}{(2d)^n \mathbb{E}[e^{\beta V}]^n} Z_n(\beta),$$

is a positive martingale. In particular, it has an almost sure limit $M^{(\beta)}$ and a simple zero-one law tells us that this limit is 0 with probability 0 or 1. Comets and Yoshida [CY06] show that there is a phase transition in the sense that there is a criti-

cal $\beta_c \in [0, \infty]$ such that

$$\begin{aligned} M^{(\beta)} &> 0 \text{ } \mathbb{P}\text{-almost surely} && \text{if } 0 < \beta < \beta_c \\ M^{(\beta)} &= 0 \text{ } \mathbb{P}\text{-almost surely} && \text{if } \beta > \beta_c \end{aligned} \quad .$$

The critical parameter depends both on the underlying dimension and the distribution of the environment. Moreover, [CY06] show that if $\beta < \beta_c$, then a central limit theorem holds, so that the polymer behaves diffusively under the polymer measure. For a precise formulation, introduce the rescaled version of the path

$$\omega^{(n)} = \left(\frac{\omega_{[nt]}}{\sqrt{n}} \right)_{t \geq 0} .$$

Theorem 1.1.1 ([CY06]). *If $\beta < \beta_c$, then for all bounded functions F on the path space,*

$$\lim_{n \rightarrow \infty} \mu_n^{(\beta)}[F(\omega^{(n)})] = \mathbf{E}F(B) ,$$

in probability, where B is a Brownian motion with diffusion matrix $d^{-1}I_d$.

This result motivates the following definition. We say that if $\beta < \beta_c$ then *weak disorder* holds, whereas the situation $\beta > \beta_c$ is referred to as the *strong disorder* regime. Under strong disorder, it is believed that the polymers behave superdiffusively at least in dimension $d = 1$, compare e.g. [IS88], but the only rigorous proofs known are for last-passage percolation, see [Joh00], which is closely related to the $\beta = \infty$ case.

However, it is possible to show that the strong disorder phase is fundamentally different from weak disorder. To formulate the results, we need to introduce a further important quantity, namely the *free energy* defined for $\beta > 0$ as

$$\varphi(\beta) = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} \log Z_n(\beta) = \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n(\beta) ,$$

where one first checks that the first limit exists, since $\mathbb{E} \log Z_n$ is superadditive and one can then use a concentration inequality to show it agrees with the same expression without expectations, for the details see e.g. [CSY03, Prop. 2.5].

By Jensen's inequality, one obtains an immediate upper bound on the free energy

$$\varphi(\beta) = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} \log Z_n(\beta) \leq \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E} Z_n(\beta) = \lambda(\beta) + \log(2d) , \quad (1.1)$$

where $\lambda(\beta) = \log \mathbb{E}[e^{\beta V}]$ is the logarithmic moment generating function of the disorder at a particular site. Naturally, one would like to know when this inequality is sharp. Again, in [CY06], it is shown that there exists $\beta_c^\varphi \in [0, \infty]$ such that

$$\begin{aligned} \varphi(\beta) &= \lambda(\beta) + \log(2d) && \text{if } \beta \leq \beta_c^\varphi \\ \varphi(\beta) &< \lambda(\beta) + \log(2d) && \text{if } \beta > \beta_c^\varphi \end{aligned} \quad .$$

In general, one can easily check that $\beta_c \leq \beta_c^\varphi$. In the regime $\beta > \beta_c^\varphi$ (which is sometimes known as very strong disorder), Carmona and Hu [CH02] and Comets,

Shiga and Yoshida [CSY03] show that polymers localize and can “feel” the effect of the environment. More precisely, if we define

$$I_n = \mu_{n-1}^{\otimes 2}(\omega_n = \tilde{\omega}_n),$$

as the probability that two polymers that are picked independently from $\mu_n^{(\beta)}$ (given the same environment) meet at the n th step, then

$$\sum_{k=1}^n I_k,$$

represents the expected collision time between two independently picked polymers. With this notation, we can formulate the following theorem giving a localization result in very strong disorder.

Theorem 1.1.2. [CH02, CSY03] *For $\beta \geq 0$,*

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n I_j > 0 \text{ almost surely} \iff \beta > \beta_c^\varphi.$$

Intuitively, this means that under very strong disorder two polymers picked independently tend to overlap a positive fraction of time. If we compare to the case when $d \geq 3$ and $\beta = 0$, then $\sum_{k=1}^n I_k$ is bounded uniformly in n , so that $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n I_n = 0$.

Now, we have seen that under certain conditions, we can either see that the environment does not have a strong effect or we can observe a localization effect. We now summarize what is known about the critical parameters. As mentioned above, we have in general that $\beta_c^\varphi \geq \beta_c$. It is conjectured that both criteria describe the same phase transition, however up to now, it has only been shown in the case $d = 1, 2$ when $\beta_c = \beta_c^\varphi = 0$. First it was shown that $\beta_c = 0$ by [CH02] for Gaussian environment and in the general case by [CSY03]. Then [CV06] show that for $d = 1$, $\beta_c^\varphi = 0$ so that both descriptions agree and finally [Lac09] shows that $\beta_c^\varphi = 0$ in dimension $d = 2$. In dimensions, $d > 3$, [Bol89] shows that $\beta_c > 0$. It is instructive to repeat his argument. Define

$$\pi_d = \mathbf{P}^{\otimes 2}\{\omega_n = \tilde{\omega}_n \text{ for some } n \geq 1\},$$

where under $\mathbf{P}^{\otimes 2}$, ω and $\tilde{\omega}$ are two independent, simple, symmetric random walks on \mathbb{Z}^d .

Proposition 1.1.3 ([Bol89]). *Suppose that $d \geq 3$ and that*

$$\lambda(2\beta) - 2\lambda(\beta) < \log(1/\pi_d), \tag{1.2}$$

then $M^{(\beta)} > 0$ \mathbb{P} -a.s.

Proof. First, note that

$$M_n^{(\beta)} = \frac{1}{(2d)^n \mathbb{E}[e^{\beta V}]^n} Z_n(\beta) = \mathbf{P}[e^{\sum_{j=1}^n \beta V(j, \omega_j) - n\lambda(\beta)}],$$

where \mathbf{P} is the law of a simple, symmetric random walk on \mathbb{Z}^d . Then, we can write

$$(M_n^{(\beta)})^2 = \mathbf{P}^{\otimes 2} [e^{\beta \sum_{j=1}^n V(j, \omega_j) - n\lambda(\beta)} e^{\beta \sum_{j=1}^n V(j, \tilde{\omega}_j) - n\lambda(\beta)}],$$

here $\mathbf{P}^{\otimes 2}[\cdot]$ denotes the expectation with respect to $\mathbf{P}^{\otimes 2}$. Hence, we can compute

$$\begin{aligned} \mathbb{E}[(M_n^{(\beta)})^2] &= \mathbb{E} \left[\mathbf{P}^{\otimes 2} \left[\prod_{j=1}^n e^{\beta(V(j, \omega_j) + V(j, \tilde{\omega}_j)) - 2\lambda(\beta)} \right] \right] \\ &= \mathbf{P}^{\otimes 2} \left[\prod_{j=1}^n (e^{\lambda(2\beta) - 2\lambda(\beta)} \mathbb{1}_{\{\omega_j = \tilde{\omega}_j\}} + \mathbb{1}_{\{\omega_j \neq \tilde{\omega}_j\}}) \right] \\ &= \mathbf{P}^{\otimes 2} [e^{(\lambda(2\beta) - 2\lambda(\beta))N_n}], \end{aligned}$$

where $N_n = \sum_{j=1}^n \mathbb{1}_{\{\omega_j = \tilde{\omega}_j\}}$ is the number of collisions between ω and $\tilde{\omega}$. As $n \rightarrow \infty$, $N_n \uparrow N_\infty$ and by monotone convergence $\mathbb{E}[(M_n^{(\beta)})^2] \uparrow \mathbf{P}^{\otimes 2}[e^{(\lambda(2\beta) - 2\lambda(\beta))N_\infty}]$. Since N_∞ is the number of visits to zero of the difference random walk $S_n = \omega_n - \tilde{\omega}_n$, N_∞ is geometrically distributed with parameter π_d . In particular, it follows that

$$\sup_{n \in \mathbb{N}} \mathbb{E}[(M_n^{(\beta)})^2] < \infty \iff \lambda(2\beta) - 2\lambda(\beta) + \log \pi_d < 0,$$

i.e. if the condition (1.2) is satisfied. So we can deduce that in that case $M_n^{(\beta)}$ is L^2 -convergent, in particular $\mathbb{E}M^{(\beta)} = \lim_{n \rightarrow \infty} \mathbb{E}M_n^{(\beta)} = 1$. Hence, $M^{(\beta)} > 0$ \mathbb{P} -a.s. \square

However, this L^2 criterion only gives a lower bound on β_c , and at least under certain conditions on the environment and the dimension this bound can be shown not be sharp, see [Bir04, BGdH08, CC09].

In principle, this leaves the option that $\beta_c = \infty$ in dimension $d \geq 3$, however following [Com05] one can show that if we define the function

$$f(\beta) = \lambda(\beta) + \log(2d) - \beta\lambda'(\beta),$$

and if f has a positive root, which we denote by β^f , then for all $\beta > 0$

$$\varphi(\beta) \leq \min \left\{ \lambda(\beta) + \log(2d), \beta \frac{\lambda(\beta^f) + \log(2d)}{\beta^f} \right\}. \quad (1.3)$$

We postpone the (easy) proof to Proposition 2.6.2. Note, however, that this condition implies by the convexity of $\lambda(\beta)$ that for $\beta > \beta^f$,

$$\varphi(\beta) < \lambda(\beta) + \log(2d).$$

In particular, $\beta_c^\mathcal{C} \leq \beta^f$, so that if f has a positive root, then we can observe a phase transition. Now, f is defined only in terms of the distribution of V , and one can check, see Lemma 2.1.2 that f has a positive root unless V is bounded from above and has an atom of mass $\geq \frac{1}{2d}$ at its essential supremum.

We will come back to the role of the function f in what follows. To complete this section, we will present an example when the environment is particular simple.

Example 1.1.4 (Bernoulli environment). We consider the case that the environment V is Bernoulli, i.e. suppose that $\mathbb{P}\{V = 1\} = p = 1 - \mathbb{P}\{V = 0\}$, $p \neq 0, 1$. If $d = 1, 2$, then we know that we are in the strong disorder phase irrespectively of p . Thus, we can concentrate on the case $d \geq 3$. Then, $\lambda(\beta) = \log((1 - p) + pe^\beta)$ and a direct computation shows that

$$\lambda(2\beta) - 2\lambda(\beta) \rightarrow -\log p \quad \text{as } \beta \rightarrow \infty.$$

Therefore, by the above discussion, we observe two different scenarios depending on the value of p . If $p > \pi_d$, then $M^{(\beta)} > 0$ for all $\beta > 0$, in other words there is no phase transition and we are always in the weak disorder phase. However, if $0 < p < \frac{1}{2d}$, then $\beta^f < \infty$, so in this case there is both a weak and a strong disorder phase.

1.1.2 The tree model

Before we present our main result in the lattice case, we will consider the mean-field model of the lattice model introduced by [DS88]. This model can also be interpreted as a branching random walk (with deterministic branching) and is also closely linked to the topic of multiplicative cascades.

For a precise description of the polymers on disordered trees, let $d \geq 2$ and T be a d -ary tree such that, starting from an initial ancestor in generation 0, the *root* ρ , each vertex has exactly d *children*. A *polymer* is a finite or infinite self-avoiding path started in the root. We write $|v|$ for the generation of a vertex v and denote by $T_n = \{v \in T : |v| = n\}$ the set of vertices in the n th generation. Each $v \in T_n$ can be identified with the unique path (v_0, v_1, \dots, v_n) of its ancestors from $v_0 = \rho$ to $v_n = v$, and thus represents a polymer of length n .

Consider a non-degenerate random variable V , which has all exponential moments, i.e.

$$\mathbb{E}[e^{\beta V}] < \infty \quad \text{for all } \beta \geq 0.$$

Then we introduce the *random disorder* $\mathcal{V} = (V(v) : v \in T)$ as a collection of independent distributed weights with the same distribution as V attached to the vertices of the tree. For a finite length polymer $v \in T_n$ we introduce the *Hamiltonian*

$$- \sum_{j=1}^n V(v_j).$$

The *polymer measure* or *finite volume Gibbs measure* $\mu_n^{(\beta)}$ on T_n is defined by

$$\mu_n^{(\beta)} = \frac{1}{Z_n(\beta)} \sum_{v \in T_n} e^{\beta \sum_{j=1}^n V(v_j)} \delta_v,$$

where $\beta > 0$ is the inverse temperature and the normalising constant $Z_n(\beta)$ is the *partition function* defined as

$$Z_n(\beta) = \sum_{v \in T_n} e^{\beta \sum_{j=1}^n V(v_j)}.$$

Polymers of infinite length can be represented as a sequence $(\xi_0, \xi_1, \xi_2, \dots)$ of vertices, such that ξ_n is a vertex in the n th generation, and moreover a child of ξ_{n-1} . Such sequences are called *rays* and the set of all rays constitute the *boundary* of the tree, denoted by ∂T . We equip the boundary ∂T with the metric $d(\xi, \eta) = \exp(-\sup\{n \geq 0 : \xi_n = \eta_n\})$, for $\xi, \eta \in \partial T$, which makes ∂T a compact metric space.

We first review some of the basic properties of the model. Roughly speaking, one should expect that the behaviour of the polymer depends on the inverse temperature parameter β in the following manner: If β is small, we are in an *entropy-dominated* regime, where the disorder has no big influence and limiting features are largely the same as in the case of a uniformly distributed polymer. For large values of β we may encounter an *energy-dominated* regime where, due to the disorder, the phase space breaks up into pieces separated by free energy barriers. Polymers then follow specific tracks with large probability, an effect often called *localization*.

As already encountered for the lattice model, the mathematical analysis of polymers on disordered trees is based on the family of martingales $(M_n^{(\beta)} : n \geq 0)$ defined by

$$M_n^{(\beta)} = e^{-n(\lambda(\beta) + \log d)} Z_n(\beta), \quad \text{for } n \geq 0,$$

where

$$\lambda(\beta) = \log \mathbb{E} e^{\beta V},$$

is the logarithmic moment generating function of V . It is easy to check that, for any $\beta \geq 0$, $(M_n^{(\beta)} : n \geq 0)$ is a martingale with respect to the filtration $\mathcal{F}_n = \sigma(V(v) : |v| \leq n)$, $n \geq 0$. Since the martingale is non-negative, its limit $M^{(\beta)} = \lim_{n \rightarrow \infty} M_n^{(\beta)}$ exists almost surely. An easy application of Kolmogorov's zero-one law shows that $\mathbb{P}\{M^{(\beta)} = 0\} \in \{0, 1\}$.

Define the function

$$f(\beta) = \lambda(\beta) + \log d - \beta \lambda'(\beta) \quad \text{for } \beta \geq 0.$$

From the strict convexity of λ , we infer that $f(\beta) < \log d$ for all $\beta > 0$. We shall check in Lemma 2.1.2 below that f has a positive root unless the law of V is bounded from above with an atom of mass $\geq \frac{1}{d}$ at its essential supremum. Let β_c be the positive root, if it exists, and $\beta_c = \infty$ otherwise. Kahane and Peyrière [KP76] and Biggins [Big77] show that

$$\begin{aligned} M^{(\beta)} &> 0 \text{ almost surely, if } \beta < \beta_c, \\ M^{(\beta)} &= 0 \text{ almost surely, if } \beta \geq \beta_c. \end{aligned}$$

In particular, they show that $\mathbb{E}[M^{(\beta)}] = 1$ if and only if $\beta < \beta_c$. In this work, we are

especially interested in the free energy, defined as

$$\varphi(\beta) = \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n(\beta).$$

Unlike in the lattice model, in the tree case it has been shown that β_c , if finite, is also the critical parameter for a change in the qualitative behaviour of the free energy. Indeed,

$$\varphi(\beta) = \begin{cases} \lambda(\beta) + \log d & \text{if } \beta \leq \beta_c, \\ \frac{\beta}{\beta_c} (\lambda(\beta_c) + \log d) & \text{if } \beta > \beta_c. \end{cases} \quad (1.4)$$

This result was stated in [DS88] and proved for a continuous time analogue. An elementary proof, based on the study of the martingales $(M_n^{(\beta)} : n \geq 0)$, can be found in [BPP93]. We observe that at the critical temperature $1/\beta_c$ the model undergoes a phase transition and, for low temperatures, it is frozen in the ground state. The two phases are often called the *weak disorder* phase ($\beta < \beta_c$), and the *strong disorder* phase ($\beta > \beta_c$). See Figure 1-2 for an illustration.

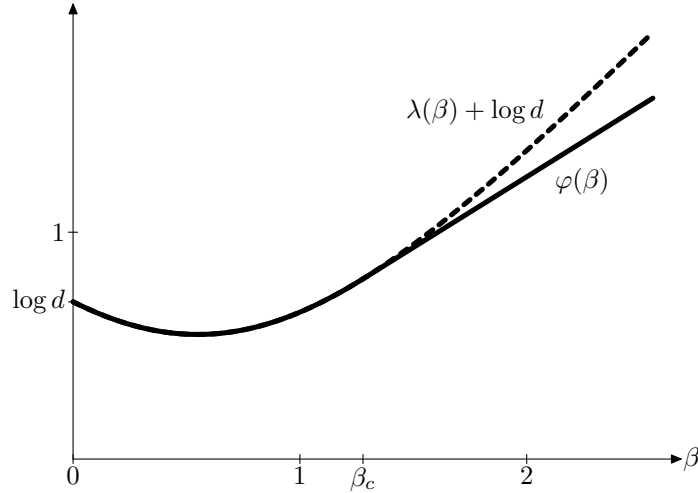


Figure 1-2: The free energy for the model when $\mathbb{P}\{V = 1\} = 1/4 = 1 - \mathbb{P}\{V = -1\}$ and $d = 2$.

In the *weak disorder phase* the form of (1.4) seems to suggest that, asymptotically, each of the d^n polymers $v \in T_n$ contributes a summand

$$\mathbb{E}[e^{\beta \sum_{j=1}^n V(v_j)}] = \exp[n\lambda(\beta)]$$

to the partition function $Z_n(\beta)$, and therefore the finite volume Gibbs measure does not localize on a significantly smaller subset of T_n . However, our first main result shows that this picture is *wrong* and already a vanishing proportion of paths make a significant contribution to the free energy. These paths can be chosen to be the vertices of a tree, which we call a *minimal supporting subtree*.

Theorem 1.1.5. *Let $0 < \beta < \beta_c$ so that we are in the weak disorder phase.*

(a) *Almost surely, there exists a tree $\tilde{T} \subset T$ of growth rate*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log |\tilde{T}_n| = f(\beta) < \log d,$$

such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{v \in \tilde{T}_n} e^{\beta \sum_{j=1}^n V(v_j)} = \varphi(\beta).$$

(b) *Almost surely for every sequence $(A_n)_{n \geq 1}$ of non-empty subsets $A_n \subset T_n$ of the vertices in the n th generation satisfying*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log |A_n| < f(\beta),$$

we have that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{v \in A_n} e^{\beta \sum_{j=1}^n V(v_j)} < \varphi(\beta).$$

Remark 1.1.6.

- Loosely speaking, if $0 < \beta < \beta_c$, vertices in generation n of the minimal supporting subtree typically contribute a summand $\exp(n\beta\lambda'(\beta))$ to the partition function $Z_n(\beta)$. As the number of such vertices is of order $\exp(nf(\beta))$, this is in line with the equation $f(\beta) + \beta\lambda'(\beta) = \varphi(\beta)$.
- The function f can be interpreted as the *entropy* of the system. Its rôle as a *multifractal spectrum* is highlighted in [Mör08].
- If we are only interested in finding sets $A_n \subset T_n$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{v \in A_n} e^{\beta \sum_{j=1}^n V(v_j)} = \varphi(\beta),$$

then the problem becomes much simpler. Indeed, if $\beta < \beta_c$, then by taking

$$A_n = \left\{ v \in T_n : \sum_{j=1}^n V(v_j) \geq \lambda'(\beta)n \right\},$$

we obtain from the Gärtner-Ellis theorem, see e.g. [dH00], that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log |A_n| = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left| \left\{ v \in T_n : \sum_{j=1}^n V(v_j) \geq \lambda'(\beta)n \right\} \right| = f(\beta),$$

so that we can deduce that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{v \in A_n} e^{\beta \sum_{j=1}^n V(v_j)} = \beta\lambda'(\beta) + f(\beta) = \varphi(\beta).$$

Hence, the difficulty lies in constructing the tree, which amounts to a consistency condition on the sets A_n .

At the critical temperature, the growth rate of the minimal supporting subtree hits zero. This suggests that in the *strong disorder phase* a subexponential set of polymers may support the free energy. This is true, and our second main result even shows that a single polymer suffices.

Theorem 1.1.7. *If $\beta_c < \infty$, then almost surely there exists a ray $\xi = (\xi_0, \xi_1, \dots) \in \partial T$ such that for any $\beta \geq \beta_c$ and sets $A_n \subset T_n$ containing the vertex ξ_n ,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{v \in A_n} e^{\beta \sum_{j=1}^n V(v_j)} = \beta \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n V(\xi_n) = \varphi(\beta).$$

Directed polymer models are intimately related to the model of ϱ -percolation introduced by Menshikov and Zuev [MZ93], which is considered for example in [KS00] and [CPV08]. Here we discuss an interesting implication of our results for this model.

To define ϱ -percolation, given an infinite, connected graph and a survival parameter $p \in (0, 1)$, we declare each edge independently to be open with probability p , or closed with probability $1 - p$. For $\varrho \in (0, 1]$ we say that ϱ -percolation occurs, if there exists an infinite self-avoiding path, along which the asymptotic proportion of open edges is at least ϱ . Our result gives a sharp criterion for the occurrence of ϱ -percolation on regular trees.

Theorem 1.1.8. *For $\varrho \in (0, 1]$,*

$$\varrho\text{-percolation occurs almost surely} \iff p \geq p_c,$$

where $p_c = \frac{1}{d}$ if $\varrho = 1$, and otherwise p_c is the unique solution in the interval $(0, \varrho)$ of the equation

$$p_c^\varrho (1 - p_c)^{1-\varrho} d = \varrho^\varrho (1 - \varrho)^{1-\varrho}.$$

Remark 1.1.9. If $\varrho = 1$, then the critical p value is $\frac{1}{d}$ which is the same as for classical percolation on a d -ary tree. However, unlike in the classical case, 1-percolation occurs at criticality. Our proofs also show that for any ϱ , in the case $p > p_c(\varrho)$ the Hausdorff dimension of the set of rays surviving ϱ -percolation agrees with that of the boundary of the surviving tree in classical percolation (with parameter p), provided the latter is non-empty.

1.1.3 Main results for the lattice model

In this section, we return to the lattice model and we investigate how much of the fairly complete picture that we developed for the tree model carries over. In particular, we are interested in describing the paths that are essential for the free energy.

To formulate our results we use the natural tree structure on the path space. More precisely, we exhibit a $2d$ -ary tree T , where the vertices in generation n are formed by the polymers of length n and the last common ancestor of two polymers is given as the longest shared initial substring, see also Figure 1-3. Then, we write $v \in T_n$ for a vertex in generation n , in other words $v = (j, \omega_j)_{j=1}^n$ for a path ω of a simple random walk of length n .

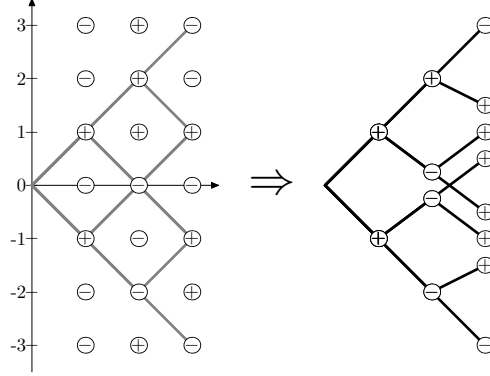


Figure 1-3: The $2d$ -ary tree embedded in the path space. Here, $d = 1$ and $V = 1$ or -1 with equal probability.

In this construction, a non-intersecting path $(v_j)_{j=1}^n$ in the tree corresponds to a (simple random walk) path $(\omega_j)_{j=1}^n$ via $v_j \leftrightarrow (\omega_i)_{i=1}^j$. With this identification, we can also think of attaching the weight $V(v) = V(n, \omega_n)$ to a vertex $v = (j, \omega_j)_{j=1}^n$. In this case, two vertices in the tree are equipped with the same weight if they correspond to two paths (on the lattice) of the same length ending up at the same point of \mathbb{Z}^d .

Thus, from the perspective of the path space, it is natural to compare the lattice model with the model from the previous section defined on a $2d$ -ary tree, which we will refer to as the mean-field model to avoid confusion. If we forget about the underlying spatial structure, we are effectively comparing a model with independent weights attached to the vertices to a model with weights given by a complicated dependency structure (inherited from the lattice). Nevertheless, one important feature that is common to both models is that vertices in different generations have independent weights.

As for the mean-field model, we are interested in finding a subtree (this time of the tree embedded in the path space) which contains the paths that are essential for the free energy. We have already seen in Remark 1.1.6, that the problem becomes easier if we only ask for the right number of paths on the n th level, but do not impose consistency restrictions on two consecutive levels. For lattice polymers, this is still a non-trivial problem, which was implicitly solved by Comets, Popov and Vachkovskaia [CPV08]. Their results are formulated in terms of the setting of ϱ -percolation, but they can also be interpreted in the directed polymer framework, as we have briefly seen at the end of Section 1.1.2. Even though all the results stated in their article are formulated for a Bernoulli environment, the proofs can be easily adapted to also hold for a general environment with exponential moments.

In order to state the theorem, we need the following lemma taken from [CPV08].

Lemma 1.1.10. *The time constants*

$$\alpha^+ = \lim_{n \rightarrow \infty} \max_{v \in T_n} \frac{1}{n} \sum_{j=1}^n V(v_j) \quad \text{and} \quad \alpha^- = \lim_{n \rightarrow \infty} \min_{v \in T_n} \frac{1}{n} \sum_{j=1}^n V(v_j),$$

are \mathbb{P} -almost surely well-defined and

$$\alpha^+ = \lim_{\beta \rightarrow \infty} \varphi(\beta)/\beta \quad \text{and} \quad \alpha^- = \lim_{\beta \rightarrow -\infty} \varphi(\beta)/\beta.$$

Proof. Clearly, for $\beta > 0$,

$$\exp \left\{ \beta \max_{v \in T_n} \sum_{j=1}^n V(v_j) \right\} \leq Z_n(\beta) \leq (2d)^n \exp \left\{ \beta \max_{v \in T_n} \sum_{j=1}^n V(v_j) \right\}.$$

Hence,

$$\frac{1}{n\beta} \log Z_n(\beta) - \frac{1}{\beta} \log(2d) \leq \max_{v \in T_n} \frac{1}{n} \sum_{j=1}^n V(v_j) \leq \frac{1}{n\beta} \log Z_n(\beta).$$

Letting $n \rightarrow \infty$ yields,

$$\frac{\varphi(\beta) - \log(2d)}{\beta} \leq \liminf_{n \rightarrow \infty} \max_{v \in T_n} \frac{1}{n} \sum_{j=1}^n V(v_j) \leq \limsup_{n \rightarrow \infty} \max_{v \in T_n} \frac{1}{n} \sum_{j=1}^n V(v_j) \leq \frac{\varphi(\beta)}{\beta}.$$

We will see in Lemma 2.6.1 that the right hand side is decreasing in β and the left hand side is increasing. Therefore, letting $\beta \rightarrow \infty$ gives $\alpha^+ = \lim_{\beta \rightarrow \infty} \frac{1}{\beta} \varphi(\beta)$ as required. Similarly, one can show that $\alpha^- = \lim_{\beta \rightarrow -\infty} \frac{1}{\beta} \varphi(\beta)$. \square

In the following theorem, the rate function is given by the Legendre-Fenchel transform of the free energy φ which is defined as

$$\varphi^*(\alpha) = \sup_{\beta \in \mathbb{R}} (\alpha\beta - \varphi(\beta)).$$

Since α^- and α^+ are by the previous lemma the slopes of the asymptotes of φ at $\pm\infty$ respectively, we have that $\varphi^*(\alpha) < \infty$ if and only if $\alpha \in [\alpha^-, \alpha^+]$.

Theorem 1.1.11 ([CPV08]). *For all $\alpha \in (\mathbb{E}V, \infty)$, $\alpha \neq \alpha^+$,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \left| \left\{ v \in T_n : \sum_{j=1}^n V(v_j) \geq \alpha n \right\} \right| = -\varphi^*(\alpha).$$

Remark 1.1.12. This result would follow fairly directly from the Gärtner-Ellis theorem, if one could show that φ^* was strictly convex on $[\alpha^-, \alpha^+]$ as it is conjectured in [CPV08]. Equivalently, one would have to show that φ was differentiable. By definition of β_c^φ , it is clear that φ is differentiable on $[0, \beta_c^\varphi)$, because for β in this range

$\varphi(\beta) = \lambda(\beta) + \log(2d)$, which is obviously differentiable. However, it seems to be a difficult problem to prove differentiability outside this range. In [CPV08], the authors by-pass this problem and manage to extend the result by proving that φ is strictly convex, see the next result, Theorem 1.1.13. Note also that the restriction $\alpha \neq \alpha^+$ could be removed, if one could show that φ^* was continuous at α^+ .

An essential step in the proof of the above theorem is showing that the free energy is strictly convex for all $\beta > 0$.

Theorem 1.1.13 ([CPV08]). *The free energy φ is a strictly convex function in β . By Legendre-Fenchel duality, φ^* is a differentiable function on the interior of (α^-, α^+) .*

The proof of this theorem relies very much on the geometry of the lattice. Namely, once one has found a single path that collects many sites where the environment is large, one can slightly perturb the path at some sites to gain many paths which see almost the same environment.

Remark 1.1.14. *Comparison with mean-field model.* Let β^f be the positive root of the function

$$f(\beta) = \lambda(\beta) + \log(2d) - \beta\lambda'(\beta),$$

and assume that β^f is finite, then we know from the previous section 1.1.2 that β^f correspond to the critical temperature in the mean-field model. Moreover, as we have seen in (1.3), c.f. also Proposition 2.6.2 later on, we can bound the free energy by

$$\varphi(\beta) \leq \min \left\{ \lambda(\beta) + \log(2d); \beta \frac{\lambda(\beta^f) + \log(2d)}{\beta^f} \right\} = \varphi^{\text{mf}}(\beta), \quad (1.5)$$

where φ^{mf} denotes the free energy in the mean field model and gives thus an upper bound on the free energy in the lattice model. However, Theorem 1.1.13 guarantees that $\beta_c^\varphi < \beta^f$, i.e. the critical parameter of the associated tree model is not critical for the lattice model. Indeed, if $\beta_c^\varphi = \beta_c^f$, then the convexity of φ and the bound from (1.5) imply that φ is linear for $\beta > \beta^f$, contradicting the strict convexity.

As an immediate corollary of Theorem 1.1.11, we can solve the problem of finding the right paths supporting the free energy when not insisting on a consistent tree structure. Define

$$\alpha(\beta) := \sup \{ \rho \geq \mathbb{E}V : (\varphi^*)'(\rho) = \beta \}.$$

Note that since φ^* is a convex and differentiable function, it follows that $\beta \mapsto \alpha(\beta)$ is a strictly increasing function on $[0, \infty)$, which is however not necessarily continuous. Moreover, for $0 < \beta < \beta_c^\varphi$, $\alpha(\beta) = \lambda'(\beta)$ by the properties of the Legendre-Fenchel transform. Also, note that if $\beta < \beta_c^\varphi$, then $-\varphi^*(\alpha(\beta)) = f(\beta)$.

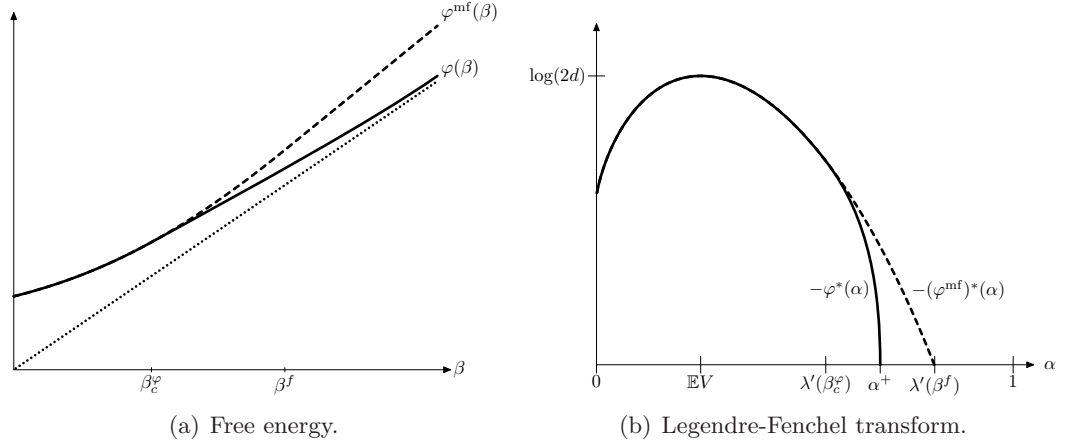


Figure 1-4: (a) A schematic representation of the free energy $\varphi(\beta)$ for the lattice model, compared to the exact free energy φ^{mf} (dashed line) of the mean-field model for V Bernoulli with success probability $p = \frac{1}{4}$ and the asymptote with slope α^+ (dotted line). (b) The Legendre-Fenchel transform φ^* associated to the free energy in the lattice model compared to the transform of the free energy $(\varphi^{\text{mf}})^*$ of the mean-field model (dashed line).

Corollary 1.1.15. *Let $\beta > 0$ and define*

$$A_n = \left\{ (j, \xi_j)_{j=1}^n : \sum_{j=1}^n V(j, \xi_j) \geq n\alpha(\beta) \right\}.$$

Then, we have that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{(j, \xi_j)_{j=1}^n \in A_n} e^{\beta \sum_{j=1}^n V(j, \xi_j)} = \varphi(\beta).$$

In the mean field model we were able to show which paths are important on the n th level, but we were also able to define a consistent set of paths that supports the free energy. Fortunately, our techniques that rely mainly on martingale arguments are sufficiently general to translate to the lattice case. Our main result states that in the weak disorder phase, even though a central limit theorem holds, we can nevertheless observe a localization on the path space. More precisely, we can find a subtree whose growth rate is strictly smaller than the full tree that supports the free energy. Moreover, any smaller subtree will not suffice.

Theorem 1.1.16. (a) Let $\beta < \beta_c$ so that we are in the weak disorder phase, then almost surely there exists a tree $\tilde{T} \subset T$ of growth rate

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log |\tilde{T}_n| = f(\beta)$$

such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{v \in \tilde{T}_n} e^{\beta \sum_{j=1}^n V(v_j)} = \varphi(\beta).$$

(b) Let $\beta > 0$. Then, almost surely for every sequence $(A_n)_{n \geq 1}$ of subsets $A_n \subset T_n$ of the vertices in the n th generation satisfying

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log |A_n| < -\varphi^*(\alpha(\beta))$$

we have that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{v \in A_n} e^{\beta \sum_{j=1}^n V(v_j)} < \varphi(\beta).$$

Remark 1.1.17. (a) Note that the first part of the theorem only holds for $\beta < \beta_c$, because the positivity of the martingale limit is essential in constructing an infinite volume Gibbs measure on the path space. The second part holds for all $\beta > 0$, because it only relies on Theorem 1.1.11.

(b) It is remarkable that for $\beta < \beta_c$, the localization effect on the path space is the same that we observed for the mean field model. However, in strong disorder, we cannot expect that a single polymer suffices. Indeed, let $\beta' > 0$ and suppose ξ is a polymer supporting the free energy at β' , so that

$$\varphi(\beta') = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{v \in T_n} e^{\beta' \sum_{j=1}^n V(v_j)} = \beta' \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n V(\xi_j) \leq \alpha^+ \beta'.$$

But, by Lemma 1.1.10, $\alpha^+ \beta \leq \varphi(\beta) \leq \alpha^+ \beta + \log(2d)$, which implies first of all $\varphi(\beta') = \alpha^+ \beta'$ and secondly that the asymptotic slope of φ is α^+ . Therefore, we can deduce from the convexity of φ that for all $\beta \geq \beta'$, $\varphi(\beta) = \alpha^+ \beta$, which contradicts the strict convexity of the free energy.

(c) If we project down from the path space onto the lattice \mathbb{Z}^{1+d} , then the minimal supporting subtree is not tree-like in the following sense. Let S be a set of infinite (simple random walk) paths on \mathbb{Z}^d and denote by S_n the set that we obtain by restricting the paths in S to the n first steps. Then, we say that S is tree-like (on the lattice), if for each $\gamma \in S_{n+1}$ we have that the restriction to the first n steps $\gamma|_n \in S_n$ and for each point (j, x) in $\mathbb{N} \times \mathbb{Z}^d$, there exists a unique path γ in S such that $\gamma(j) = x$. Then for any $\beta > 0$, such a set S does not support the free energy, since $|S_n| \leq Cn^d$ and part (b) of the previous theorem says that we need at least exponential growth for a supporting subtree.

1.2 Ageing in the parabolic Anderson model

1.2.1 Motivation and overview

The long term dynamics of disordered complex systems out of equilibrium have been the subject of great interest in the past decade. A key paradigm in this research programme is the notion of ageing. Roughly speaking, in an ageing system the probability that there is no essential change of the state between time t and time $t + s(t)$ is of constant order for a period $s(t)$ which depends increasingly, and often linearly, on the time t . Hence, as time goes on, in an ageing system changes become less likely and the typical time scales of the system are increasing. Therefore, as pointed out in [BF05], ageing can be associated to the existence of infinitely many time-scales that are inherently relevant to the system. In that respect, ageing systems are distinct from metastable systems, which are characterized by a finite number of well separated time-scales, corresponding to the lifetimes of different metastable states.

Ageing systems are typically rather difficult to analyse analytically. Most results to date concern either the Langevin dynamics of relatively simple mean field spin glasses, see e.g. [BADG01], or phenomenological models like the class of *trap models*, see e.g. [Bou92, Čer06, GMW09]. The idea behind the latter is to represent a physical system as a particle moving in a random energy landscape with infinitely many valleys, or traps. Given the landscape, the particle moves according to a continuous time random walk remaining at each trap for an exponential time with a rate proportional to its depth. While there is good experimental evidence for the claim that trap models capture the dynamical behaviour of many more complex systems, a rigorous mathematical derivation of this fact exists only in very few cases.

In the present work we show that the parabolic Anderson model exhibits ageing behaviour, at least if the underlying random potential is sufficiently heavy-tailed. As a lattice model with random disorder the parabolic Anderson model is a model of significant complexity, but its linearity and strong localization features make it considerably easier to study than, for example, the dynamics of most non-mean field spin glass models.

Our work has led to three main results. The first one, Theorem 1.2.1, shows that the probability that during the time window $[t, t + \theta t]$ the profiles of the solution of the parabolic Anderson problem remain within distance $\varepsilon > 0$ of each other converges to a constant $I(\theta)$, which is strictly between zero and one. This shows that ageing holds on a linear time scale. Our second main result, Theorem 1.2.3, is an almost sure ageing result. We define a function $R(t)$ which characterizes the waiting time starting from time t until the profile changes again. We determine the precise almost sure upper envelope of $R(t)$ in terms of an integral test. The third main result, Theorem 1.2.6, is a functional scaling limit theorem for the location of the peak, which determines the profile, and for the growth rate of the solution. We give the precise statements of the results in Section 1.2.2, and in Section 1.2.3 we provide a rough guide to the proofs.

1.2.2 Statement of the main results

The parabolic Anderson model is given by the heat equation on the lattice \mathbb{Z}^d with a random potential, i.e. we consider the solution $u: (0, \infty) \times \mathbb{Z}^d \rightarrow [0, \infty)$ of the Cauchy problem

$$\begin{aligned} \frac{\partial}{\partial t} u(t, z) &= \Delta u(t, z) + \xi(z)u(t, z), & \text{for } (t, z) \in (0, \infty) \times \mathbb{Z}^d, \\ \lim_{t \downarrow 0} u(t, z) &= \mathbb{I}_0(z), & \text{for } z \in \mathbb{Z}^d. \end{aligned}$$

Here Δ is the discrete Laplacian

$$\Delta f(x) = \sum_{\substack{y \in \mathbb{Z}^d \\ y \sim x}} (f(y) - f(x)),$$

and $y \sim x$ means that y is a nearest-neighbour of site x . The potential $\xi = (\xi(z): z \in \mathbb{Z}^d)$ is a collection of independent, identically distributed random variables, which we assume to be Pareto-distributed for some $\alpha > d$, i.e.

$$\text{Prob}\{\xi(z) \leq x\} = 1 - x^{-\alpha}, \quad \text{for } x \geq 1.$$

The condition $\alpha > d$ is necessary and sufficient for the Cauchy problem to have a unique, non-negative solution, see [GM90]. We write

$$U(t) = \sum_{z \in \mathbb{Z}^d} u(t, z) \quad \text{for } t \geq 0,$$

for the *total mass* of the solution (which is finite at all times) and

$$v(t, z) = \frac{u(t, z)}{U(t)} \quad \text{for } t \geq 0, z \in \mathbb{Z}^d,$$

for its *profile*. It is not hard to see that the total mass grows superexponentially in time. Our interest is therefore focused on the changes in the profile of the solution.

Ageing: a weak limit theorem

Our first ageing result is a weak limit result. We show that for an observation window whose size is growing linearly in time, the probability of seeing no change during the window converges to a nontrivial value. The same limit is obtained when only the states at the endpoints of the observation window are considered.

Theorem 1.2.1. *For any $\theta > 0$ there exists $I(\theta) > 0$ such that, for all sufficiently small $\varepsilon > 0$,*

$$\begin{aligned} \lim_{t \rightarrow \infty} \text{Prob} \left\{ \sup_{z \in \mathbb{R}^d} \sup_{s \in [t, t+t\theta]} |v(t, z) - v(s, z)| < \varepsilon \right\} \\ = \lim_{t \rightarrow \infty} \text{Prob} \left\{ \sup_{z \in \mathbb{R}^d} |v(t, z) - v(t + t\theta, z)| < \varepsilon \right\} \\ = I(\theta). \end{aligned}$$

Remark 1.2.2.

- Note that we only have one ageing regime, which is contrast to the behaviour of the unsymmetric trap models described in [BAČ05]
- An integral representation of $I(\theta)$ will be given in Proposition 3.1.4, which shows that the limit is not derived from the generalized arcsine law as in the universal scheme for trap models described in [BAČ08]. In Proposition 3.1.5, we show that there are positive constants C_0, C_1 such that

$$\lim_{\theta \downarrow 0} \theta^{-1} (1 - I(\theta)) = C_0 \quad \text{and} \quad \lim_{\theta \uparrow \infty} \theta^d I(\theta) = C_1.$$

Ageing: an almost-sure limit theorem

The crucial ingredient in our ageing result is the fact that in the case of Pareto distributed potentials the profile of the solution of the parabolic Anderson problem can be essentially described by one parameter, the location of its *peak*. This is due to the one-point localization theorem [KLMS09, Theorem 1.2] which states that, for any \mathbb{Z}^d -valued process $(X_t : t \geq 0)$ with the property that $v(t, X_t)$ is the maximum value of the profile at time t , we have

$$v(t, X_t) \rightarrow 1 \text{ in probability.} \tag{1.6}$$

In other words, asymptotically the profile becomes completely localized in its peak. Assume for definiteness that $t \mapsto X_t$ is right-continuous and define the *residual lifetime* function by $R(t) = \sup\{s \geq 0 : X_t = X_{t+s}\}$, for $t \geq 0$. Roughly speaking, $R(t)$ is the waiting time, at time t , until the next change of peak, see the schematic picture in Figure 1-5. We have shown in Theorem 1.2.1 that the law of $R(t)/t$ converges to the law given by the distribution function $1 - I$. In the following theorem, we describe the smallest asymptotic upper envelope for the process $(R(t) : t \geq 0)$.

Theorem 1.2.3 (Almost sure ageing). *For any nondecreasing function $h : (0, \infty) \rightarrow (0, \infty)$ we have, almost surely,*

$$\limsup_{t \rightarrow \infty} \frac{R(t)}{th(t)} = \begin{cases} 0 & \text{if } \int_1^\infty \frac{dt}{th(t)^d} < \infty, \\ \infty & \text{if } \int_1^\infty \frac{dt}{th(t)^d} = \infty. \end{cases}$$

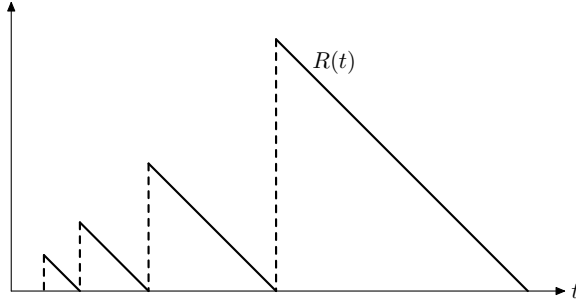


Figure 1-5: A schematic representation of the residual lifetime function R .

A functional scaling limit theorem

To complete the discussion of the temporal behaviour of the solution it is natural to look for a functional limit theorem under suitable space-time scaling of the solution. From [HMS08, Theorem 1.2] we know that there are heavy fluctuations even in the *logarithm* of the total mass, as we have for $t \uparrow \infty$,

$$\frac{(\log t)^{\frac{d}{\alpha-d}}}{t^{\frac{\alpha}{\alpha-d}}} \log U(t) \Rightarrow Y, \quad (1.7)$$

where Y is a random variable of extremal Fréchet type with shape parameter $\alpha - d$. We therefore focus on the profile of the solution and interpret it as giving rise to a probability measure on \mathbb{R}^d , by considering

$$\sum_{x \in \mathbb{Z}^d} v(tT, x) \delta\left(\left(\frac{\log T}{T}\right)^{\frac{\alpha}{\alpha-d}} x\right),$$

where $\delta(x)$ is a Dirac mass in the point $x \in \mathbb{R}^d$. By construction, this point measure is an element of the space $\mathcal{M}(\mathbb{R}^d)$ of probability measures on \mathbb{R}^d that we equip with the weak topology.

Proposition 1.2.4 (Convergence of the scaled profile to a wandering point mass). *There exists a nondegenerate stochastic process $(Y_t : t > 0)$ such that, as $T \uparrow \infty$, the following functional scaling limit holds,*

$$\left(\sum_{x \in \mathbb{Z}^d} v(tT, x) \delta\left(\left(\frac{\log T}{T}\right)^{\frac{\alpha}{\alpha-d}} x\right) : t > 0 \right) \Rightarrow \left(\delta(Y_t) : t > 0 \right), \quad (1.8)$$

in the sense of convergence of finite dimensional distributions on the space $\mathcal{M}(\mathbb{R}^d)$ equipped with the weak topology.

Remark 1.2.5. The process $(Y_t : t > 0)$ will be described explicitly in and after Remark 1.2.7 (iii).

In this formulation of a scaling limit theorem the mode of convergence is not optimal. Also, under the given scaling, islands of diameter $o((\frac{t}{\log t})^{\frac{\alpha}{\alpha-d}})$ at time t would still be

mapped onto single points, and hence the spatial scaling is not sensitive to the one-point localization described in the previous section. We now state an optimal result in the form of a functional scaling limit theorem in the Skorokhod topology for the localization point itself. Additionally, we prove joint convergence of the localization point together with the value of the potential there. This leads to a Markovian limit process which is easier to describe, and from which the non-Markovian process $(Y_t : t > 0)$ can be derived by projection. This approach also yields an extension of (1.7) to a functional limit theorem. Here and in the following we denote by $|x|$ the ℓ^1 -norm of $x \in \mathbb{R}^d$.

Theorem 1.2.6 (Functional scaling limit theorem).

There exists a time-inhomogeneous Markov process $((Y_t^{(1)}, Y_t^{(2)}) : t > 0)$ on $\mathbb{R}^d \times \mathbb{R}$ such that,

(a) *as $T \rightarrow \infty$, we have*

$$\left(\left(\left(\frac{\log T}{T} \right)^{\frac{\alpha}{\alpha-d}} X_{tT}, \left(\frac{\log T}{T} \right)^{\frac{d}{\alpha-d}} \xi(X_{tT}) \right) : t > 0 \right) \Rightarrow \left((Y_t^{(1)}, Y_t^{(2)} + \frac{d}{\alpha-d} |Y_t^{(1)}|) : t > 0 \right),$$

in distribution on the space $D(0, \infty)$ of càdlàg functions $f : (0, \infty) \rightarrow \mathbb{R}^d \times \mathbb{R}$ with respect to the Skorokhod topology on compact subintervals;

(b) *as $T \rightarrow \infty$, we have*

$$\left(\left(\frac{\log T}{T} \right)^{\frac{d}{\alpha-d}} \frac{\log U(tT)}{tT} : t > 0 \right) \Rightarrow \left(Y_t^{(2)} + \frac{d}{\alpha-d} \left(1 - \frac{1}{t}\right) |Y_t^{(1)}| : t > 0 \right),$$

in distribution on the space $C(0, \infty)$ of continuous functions $f : (0, \infty) \rightarrow \mathbb{R}$ with respect to the uniform topology on compact subintervals.

Remark 1.2.7.

- (i) Projecting the process onto the first component at time $t = 1$ we recover the result of [KLMS09, Theorem 1.3]. This result shows in particular that the peak X_t of the profile escapes with superlinear speed.
- (ii) From the proof of this result it is easy to see that the convergence in both parts of Theorem 1.2.6 also holds simultaneously on the space of càdlàg functions $f : (0, \infty) \rightarrow \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$ with respect to the Skorokhod topology on compact subintervals.
- (iii) The process $(Y_t : t > 0)$ in Proposition 1.2.4 is equal to the projected process $(Y_t^{(1)} : t > 0)$.

In order to describe the limit process we need to introduce some notation. Denote by Π a Poisson point process on $H^0 = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : y > -\frac{d}{\alpha-d} |x|\}$ with intensity measure

$$\nu(dx, dy) = dx \otimes \frac{\alpha dy}{(y + \frac{d}{\alpha-d} |x|)^{\alpha+1}}.$$

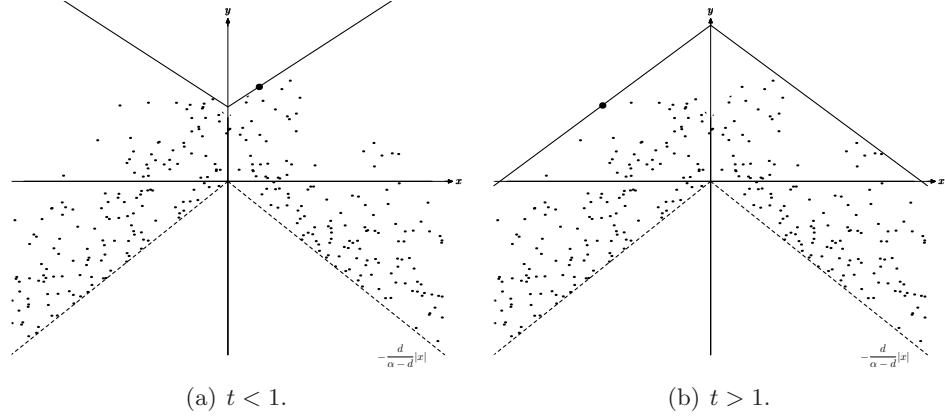


Figure 1-6: The definition of the process $(Y_t^{(1)}, Y_t^{(2)})$ in terms of the point process Π . Note that t parametrizes the opening angle of the cone, see (a) for $t < 1$ and (b) for $t > 1$.

Given the point process, we can define an \mathbb{R}^d -valued process $Y_t^{(1)}$ and an \mathbb{R} -valued process $Y_t^{(2)}$ in the following way. Fix $t > 0$ and define the open cone with tip $(0, z)$

$$\mathcal{C}_t(z) = \left\{ (x, y) \in \mathbb{R}^d \times \mathbb{R} : y + \frac{d}{\alpha-d} \left(1 - \frac{1}{t}\right) |x| > z \right\},$$

and let

$$\mathcal{C}_t = \text{cl} \left(\bigcup_{\substack{z > 0 \\ \Pi(\mathcal{C}_t(z)) = 0}} \mathcal{C}_t(z) \right).$$

Informally, \mathcal{C}_t is the closure of the first cone $\mathcal{C}_t(z)$ that ‘touches’ the point process as we decrease z from infinity. Since $\mathcal{C}_t \cap \Pi$ contains at most two points, we can define $(Y_t^{(1)}, Y_t^{(2)})$ as the point in this intersection whose projection on the first component has the largest ℓ^1 -norm, see Figures 1-6(a) and 1-6(b) for an illustration. The resulting process $((Y_t^{(1)}, Y_t^{(2)}): t > 0)$ is an element of $D(0, \infty)$.

The derived processes in Theorem 1.2.6 can be described as follows:

- $((Y_t^{(2)} + \frac{d}{\alpha-d} |Y_t^{(1)}|): t > 0)$ corresponds to the *vertical distance* of the point $(Y_t^{(1)}, Y_t^{(2)})$ to the boundary of the domain given by the curve $y = -\frac{d}{\alpha-d} |x|$;
- $((Y_t^{(2)} + (1 - \frac{1}{t}) |Y_t^{(1)}|): t > 0)$ corresponds to the *y-coordinate of the tip* of the cone \mathcal{C}_t .

Remark 1.2.8. *Time evolution of the process.*

$(Y_1^{(1)}, Y_1^{(2)})$ is the ‘highest’ point of the Poisson point process Π . Given $(Y_t^{(1)}, Y_t^{(2)})$ and $s \geq t$ we consider the surface given by all $(x, y) \in \mathbb{R}^d \times \mathbb{R}$ such that

$$y = Y_t^{(2)} - \frac{d}{\alpha-d} \left(1 - \frac{1}{s}\right) (|x| - |Y_t^{(1)}|).$$

For $s = t$ there are no points of Π above this surface, while $(Y_t^{(1)}, Y_t^{(2)})$ (and possibly one further point) is lying on it. We now increase the parameter s until the

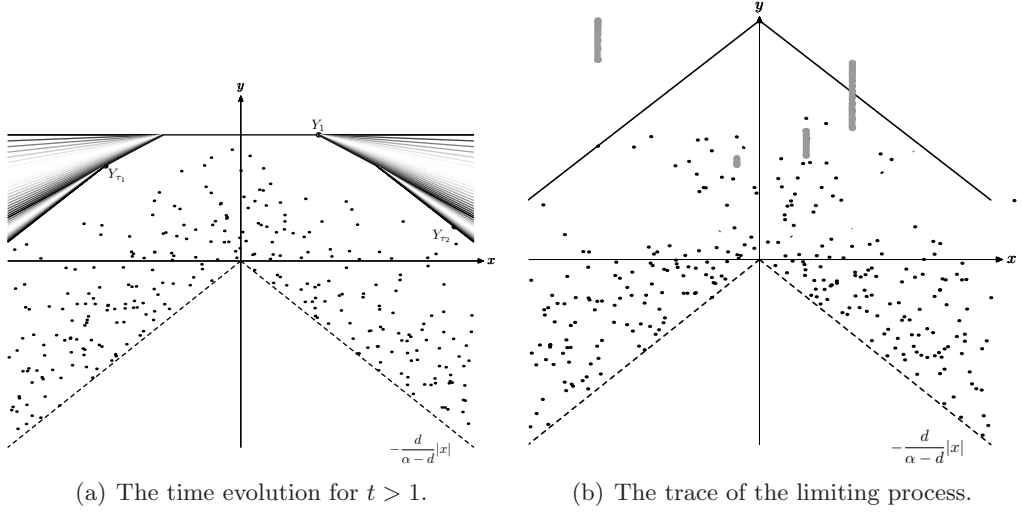


Figure 1-7: (a) The time evolution for $t > 1$ of the jump process $Y_t = (Y_t^{(1)}, Y_t^{(2)})$ (large dots), defined in terms of the Poisson point process (small dots). (b) The trace of the limiting process $(Y_t^{(1)}, Y_t^{(2)} + \frac{d}{\alpha-d}(1 - \frac{1}{t})|Y_t^{(1)}|)$. The second component corresponds to the y -coordinate of the tip of the defining cone.

surface hits a further point of Π . At this time $s > t$ the process jumps to this new point $(Y_s^{(1)}, Y_s^{(2)})$. Geometrically, increasing s means opening the cone further, while keeping the point $(Y_t^{(1)}, Y_t^{(2)})$ on the boundary and moving the tip upwards on the y -axis, see Figure 1-7(a). Similarly, given the point $(Y_t^{(1)}, Y_t^{(2)})$ one can go backwards in time by decreasing s , or equivalently closing the cone and moving the tip downwards on the y -axis. The general independence properties of Poisson processes ensure that this procedure yields a process $((Y_t^{(1)}, Y_t^{(2)}): t > 0)$ which is Markovian in both the forward and backward direction. The process $(Y_t^{(2)} + \frac{d}{\alpha-d}(1 - \frac{1}{t})|Y_t^{(1)}|: t > 0)$ is continuous, which can be seen directly from its interpretation as the y -coordinate of the tip of the cone, see Figure 1-7(b).

1.2.3 Strategy of the proofs and overview

It is shown in [KLMS09] that, almost surely, for all large t the total mass $U(t)$ can be approximated by a variational problem. To understand the motivation behind this variational problem, recall first that the solution u can be written using the (time-reversed) Feynman-Kac formula as

$$u(t, z) = \mathbf{E}_0 \left[\exp \left\{ \int_0^t \xi(W_s) ds \right\} \mathbb{I}_{\{W_t = z\}} \right],$$

where $(W_s, s \geq 0)$ (under \mathbf{P}_0) is a continuous-time simple random walk on \mathbb{Z}^d with generator Δ starting at 0. Therefore, the total mass can be expressed as

$$U(t) = \mathbf{E}_0 \left[\exp \left\{ \int_0^t \xi(W_s) ds \right\} \right],$$

Heuristically, for fixed $t > 0$, the paths $(W_s : 0 \leq s \leq t)$ that have the biggest influence are those that spend most of their time at a site z that has a large potential $\xi(z)$ and is not too far away from the origin, so that the random walk can reach it by time t with a reasonable probability.

To formalize this idea, let $A_t^{z,\rho}$, $\rho \in (0, 1)$ be the strategy for the random walk W to wander to z during $[0, \rho t)$ and to stay there until time t . Then, for $|z| \gg t$,

$$\mathbf{P}_0(A_t^{z,\rho}) \approx \exp \left\{ -|z| \log \frac{|z|}{e\rho t} + \eta(z) \right\},$$

where $|z|$ is the ℓ^1 -norm on \mathbb{R}^d and $\eta(z)$ is the logarithm of the number of paths of length $|z|$ leading from 0 to z . Hence, we obtain

$$\frac{1}{t} \log U(t) \gtrsim \sup_{z \in \mathbb{Z}^d} \sup_{\rho \in (0,1)} \left[(1-\rho)\xi(z) - \frac{|z|}{t} \log \frac{|z|}{e\rho t} + \frac{\eta(z)}{t} \right].$$

Now, we notice that the optimal $\rho \approx |z|/(t\xi(z))$, so that it becomes plausible that

$$\frac{1}{t} \log U(t) \sim \max_{z \in \mathbb{Z}^d} \Phi_t(z), \quad (1.9)$$

where, for any $t \geq 0$, the functional Φ_t is defined as

$$\Phi_t(z) = \xi(z) - \frac{|z|}{t} \log \xi(z) + \frac{\eta(z)}{t},$$

for $z \in \mathbb{Z}^d$ with $t\xi(z) \geq |z|$, and $\Phi_t(z) = 0$ for other values of z . Making this calculation rigorous requires a very detailed analysis of the Feynman-Kac formula.

[KLMS09] continue their analysis by showing that the peak X_t of the profile agrees for most times t with the maximizer Z_t of the functional Φ_t . This maximizer is uniquely defined, if we impose the condition that $t \mapsto Z_t$ is right-continuous. Defining the two scaling functions

$$r_t = \left(\frac{t}{\log t} \right)^{\frac{\alpha}{\alpha-d}} \quad \text{and} \quad a_t = \left(\frac{t}{\log t} \right)^{\frac{d}{\alpha-d}},$$

it is shown in [KLMS09] (refining the argument of [HMS08]) that, as $t \rightarrow \infty$, the point process

$$\Pi_t = \sum_{\substack{z \in \mathbb{Z}^d \\ t\xi(z) \geq |z|}} \delta_{\left(\frac{z}{r_t}, \frac{\Phi_t(z)}{a_t} \right)} \quad (1.10)$$

converges (in a suitable sense) to the Poisson point process Π on H_0 defined above.

The ‘annealed’ ageing result, Theorem 1.2.1, is proved in Section 3.1.1. We show in Lemma 3.1.9 that

$$\begin{aligned} \lim_{t \rightarrow \infty} \text{Prob} \left\{ \sup_{z \in \mathbb{R}^d} \sup_{s \in [t, t+\theta]} |v(t, z) - v(s, z)| < \varepsilon \right\} \\ = \lim_{t \rightarrow \infty} \text{Prob} \{ Z_t = Z_{t+\theta} \}. \end{aligned}$$

The task is now to approximate the probability on the right hand side in terms of the point process Π_t . We are able to write

$$\frac{\Phi_{t+\theta t}(z)}{a_t} = \frac{\Phi_t(z)}{a_t} + \frac{\theta}{1+\theta} \frac{d}{\alpha-d} \frac{|z|}{r_t} + \text{error}, \quad (1.11)$$

where the error can be suitably controlled, see Lemma 3.1.3. Hence (in symbolic notation)

$$\begin{aligned} \text{Prob} \{ Z_t = Z_{t+\theta} \} \\ \approx \iint \text{Prob} \{ \Pi_t(dx, dy) > 0, \Pi_t\{(\bar{x}, \bar{y}) : \bar{y} > y\} = 0, \\ \Pi_t\{(\bar{x}, \bar{y}) : |\bar{x}| > |x| \text{ and } \bar{y} > y - \frac{d}{\alpha-d} \frac{\theta}{1+\theta} (|\bar{x}| - |x|)\} = 0 \}, \end{aligned}$$

where the first line of conditions on the right means that x is a maximizer of Φ_t with maximum y , and the second line means that x is also a maximizer of $\Phi_{t+\theta t}$. As $t \uparrow \infty$ the point process Π_t is replaced by Π and we can evaluate the probability.

The ‘quenched’ ageing result, Theorem 1.2.3, is proved in Section 3.2.2. We now have to consider events

$$\text{Prob} \left\{ \frac{R(t)}{t} \geq \theta_t \right\} \approx \text{Prob} \{ Z_t = Z_{t+\theta t} \},$$

for $\theta_t \uparrow \infty$. We have to significantly refine the argument above and replace the convergence of $\text{Prob} \{ Z_t = Z_{t+\theta t} \}$ by a moderate deviation statement. Indeed, for $\theta_t \uparrow \infty$ not too fast we show that

$$\text{Prob} \{ Z_t = Z_{t+\theta t} \} \sim C \theta_t^{-d},$$

for a suitable constant $C > 0$, see Proposition 3.2.1. Then, if $\varphi(t) = th(t)$, this allows us to show that the series $\sum_n \text{Prob} \{ R(e^n) \geq \varphi(e^n) \}$ converges if $\sum_n h(e^n)^{-d}$ converges, which is essentially equivalent to $\int h(t)^{-d} dt/t < \infty$. By Borel-Cantelli we get that

$$\limsup_{n \rightarrow \infty} \frac{R(e^n)}{\varphi(e^n)} = 0,$$

which implies the upper bound in Theorem 1.2.3, and the lower bound follows similarly using a slightly more delicate second moment estimate, see Lemma 3.2.5.

The proofs of the scaling limit theorems, Proposition 1.2.4 and Theorem 1.2.6 are given

in Section 3.3. By (1.11) we can describe Z_{tT} approximately as the maximizer of

$$\frac{\Phi_T(z)}{a_T} + \frac{d}{\alpha - d} \left(1 - \frac{1}{t}\right) \frac{|z|}{r_T}.$$

Instead of attacking the proof of Theorem 1.2.6 directly, we will first show a limit theorem for

$$\left(\left(\frac{Z_{tT}}{r_T}, \frac{\Phi_{tT}}{a_T} \right) : t > 0 \right). \quad (1.12)$$

Informally, we obtain

$$\begin{aligned} P \left\{ \frac{Z_{tT}}{r_T} \in A, \frac{\Phi_{tT}(Z_{tT})}{a_T} \in B \right\} \\ \approx \iint_{\substack{x \in A, \\ y + q(1 - \frac{1}{t})|x| \in B}} \text{Prob} \left\{ \Pi_T(dx dy) > 0, \right. \end{aligned}$$

$$\left. \Pi_T \{ (\bar{x}, \bar{y}) : \bar{y} - y > \frac{d}{\alpha - d} \left(1 - \frac{1}{t}\right) (|x| - |\bar{x}|) \} = 0 \right\},$$

where the first line of conditions on the right means that there is a site $z \in \mathbb{Z}^d$ such that $x = z/r_T \in A$ and $y = \Phi_T(z)/a_T \in B - q(1 - \frac{1}{t})|x|$, and the second line means that $\Phi_{tT}(z)$ is not surpassed by $\Phi_{tT}(\bar{z})$ for any other site $\bar{z} \in \mathbb{Z}^d$ with $\bar{x} = \bar{z}/r_T$. We can then use the convergence of Π_T to Π inside the formula to give a limit theorem for the one-dimensional distributions of (1.12). Checking a tightness criterion in Skorokhod space completes the argument, to show that

$$\left(\left(\frac{Z_{tT}}{r_T}, \frac{\Phi_{tT}}{a_T} \right) : t > 0 \right) \Rightarrow \left((Y_t^{(1)}, Y_t^{(2)} + \frac{d}{\alpha - d} (1 - \frac{1}{t}) |Y_t^{(1)}|) : t > 0 \right).$$

Then, Theorem 1.2.6 (b) follows using (1.9) and projecting on the second component. Observe that the convergence in (b) automatically holds in the uniform sense, as all involved processes are continuous. We note further that

$$\frac{\xi(z)}{a_T} = \frac{\Phi_T(z)}{a_T} + \frac{d}{\alpha - d} \frac{|z|}{r_T} + \text{error}.$$

This allows us to deduce Theorem 1.2.6 (a). Finally, Proposition 1.2.4 is an easy consequence of Theorem 1.2.6 (a).

Chapter 2

Directed polymers in random environment

This chapter is based on joint work with P. Mörters.¹

This chapter is structured as follows. We first concentrate on the mean field model of polymers on regular trees. In Section 2.1 we review some of the basic properties of the function f . In Section 2.2 we focus on the weak disorder phase and develop some basic ergodic theory of weighted trees, which enables us to construct and explore some properties of the infinite volume Gibbs measures. We also give an estimate on the number of polymers of length n for which the Hamiltonian is unusually small in terms of a coarse multifractal spectrum. Using this, we prove Theorem 1.1.5 in Section 2.3. More subtle techniques are required to discuss the critical case and tackle Theorem 1.1.7. These are developed in Section 2.4. In the last section on the mean-field model, Section 2.5, we translate our results to the model of ϱ -percolation and complete the proof of Theorem 1.1.8. Finally, we return to the lattice based model in Section 2.6 and first prove some general properties of the free energy and finally adapt the theory developed for the tree model to the new setting.

2.1 Preliminaries

In this section, we review some of the properties of the function f . In particular, we establish a necessary and sufficient condition for f to have a positive root, see Lemma 2.1.2. We also prove a result about the minimum of the Hamiltonian taken over the vertices in the n th generation, see Lemma 2.1.3. We start this section with a particular form of the *Laplace principle*.

Lemma 2.1.1. *As $\beta \rightarrow \infty$,*

$$\lambda'(\beta) = \frac{\mathbb{E}[V e^{\beta V}]}{\mathbb{E}[e^{\beta V}]} \rightarrow \text{ess sup } V.$$

¹part of the work regarding the mean field model has been published as [MO08].

Proof. First of all, for non-degenerate V , the function $\lambda'(\cdot)$ is strictly increasing on $[0, \infty)$, since $\lambda''(\beta) > 0$. Therefore $\lim_{\beta \rightarrow \infty} \lambda'(\beta) \in (\mathbb{E}V, \infty]$ exists. Clearly, $\lambda'(\beta) \leq \text{ess sup } V$. Therefore, it remains to show the reversed inequality. So fix a k such that $\text{ess inf } V < k < \text{ess sup } V$, then

$$\mathbb{E}[Ve^{\beta V}] \geq e^{\beta k} \left[k\mathbb{E}[\mathbb{I}\{V \geq k\}e^{\beta(V-k)}] + \mathbb{E}[\mathbb{I}\{V < k\}Ve^{\beta(V-k)}] \right].$$

Similarly,

$$\mathbb{E}[e^{\beta V}] \leq e^{\beta k} \left[\mathbb{E}[\mathbb{I}\{V \geq k\}e^{\beta(V-k)}] + 1 \right].$$

Therefore, combining the previous two inequalities we obtain

$$\lambda'(\beta) \geq \frac{k\mathbb{E}[\mathbb{I}\{V \geq k\}e^{\beta(V-k)}] + \mathbb{E}[\mathbb{I}\{V < k\}Ve^{\beta(V-k)}]}{\mathbb{E}[\mathbb{I}\{V \geq k\}e^{\beta(V-k)}] + 1}.$$

Clearly, $|\mathbb{E}[\mathbb{I}\{V < k\}Ve^{\beta(V-k)}]| \leq \mathbb{E}|V| < \infty$. So in order to show that $\lim_{\beta \rightarrow \infty} \lambda'(\beta) \geq k$ it suffices to show that the denominator in the last display diverges to infinity. Now take $\varepsilon > 0$ small enough such that $k + \varepsilon < \text{ess sup } V$, then, as $\beta \rightarrow \infty$,

$$\mathbb{E}[\mathbb{I}\{V \geq k\}e^{\beta(V-k)}] \geq e^{\beta\varepsilon} \mathbb{P}\{V > k + \varepsilon\} + \mathbb{P}\{k \leq V \leq k + \varepsilon\} \rightarrow \infty$$

Letting $k \rightarrow \text{ess sup } V$, we see that $\lim_{\beta \rightarrow \infty} \lambda'(\beta) \geq \text{ess sup } V$, which completes the proof. \square

We require the Legendre-Fenchel transform λ^* of λ defined as

$$\lambda^*(\alpha) = \sup_{\beta \in \mathbb{R}} \{\alpha\beta - \lambda(\beta)\},$$

see Figure 2.1 for an illustration.

The next result, which can be found in [Com05], gives us a necessary and sufficient condition for f to have a positive root.

Lemma 2.1.2. *f has a positive root if and only if*

- either V is unbounded,
- or $w := \text{ess sup } V$ is finite and $\mathbb{P}\{V = w\} < \frac{1}{d}$.

Proof. Using the Legendre-Fenchel transform, we find that

$$f(\beta) = \log d + \lambda(\beta) - \beta\lambda'(\beta) = \log d - \lambda^*(\lambda'(\beta)). \quad (2.1)$$

Since $f(0) = \log d$ and f is strictly decreasing and continuous, f has a positive root if and only if $\lim_{\beta \rightarrow \infty} f(\beta) < 0$. By Lemma 2.1.1,

$$\lambda'(\beta) = \frac{\mathbb{E}[Ve^{\beta V}]}{\mathbb{E}[e^{\beta V}]} \rightarrow \text{ess sup } V.$$

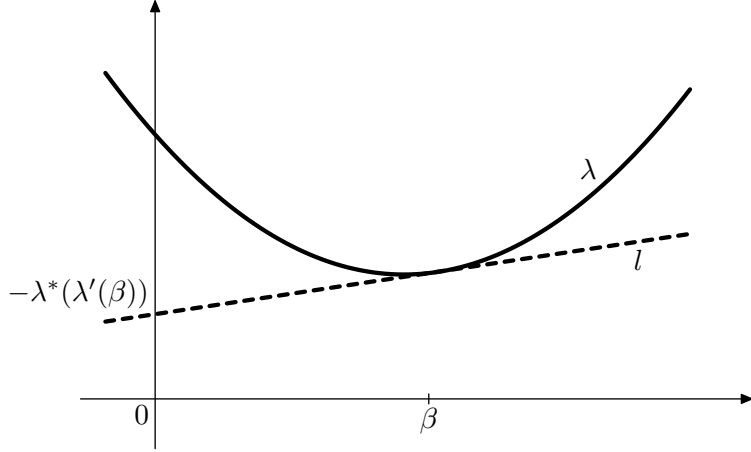


Figure 2-1: The *Legendre-Fenchel transform* of λ . Let $\alpha \in \mathbb{R}$. If l is the unique line of support of λ at β with slope α , then $-\lambda^*(\alpha)$ is equal to the y -coordinate of the intersection point of l with the vertical axis.

Therefore, if $\text{ess sup } V = \infty$, then $\lambda'(\beta) \rightarrow \infty$, which implies that $f(\beta) \rightarrow -\infty$, so that f has a positive root.

Now suppose that $w := \text{ess sup } V < \infty$. Using $\lambda'(\beta) \rightarrow w$ and the lower semi-continuity of λ^* ,

$$\begin{aligned} \lim_{\beta \rightarrow \infty} \lambda^*(\lambda'(\beta)) &= \lambda^*(w) = \sup_{\beta} (\beta w - \log \mathbb{E}[e^{\beta V}]) \\ &= -\inf_{\beta} (\log(\mathbb{P}\{V = w\} + \mathbb{E}[\mathbb{I}\{V < w\} e^{\beta(V-w)}])) = -\log \mathbb{P}\{V = w\}. \end{aligned}$$

So in particular, by (2.1), $\lim_{\beta \rightarrow \infty} f(\beta) = \log d + \log \mathbb{P}\{V = w\}$. Therefore, if $\mathbb{P}\{V = w\} < \frac{1}{d}$, then $\lim_{\beta \rightarrow \infty} f(\beta) < 0$, i.e. f has a positive root. Conversely, if $\mathbb{P}\{V = w\} \geq \frac{1}{d}$, then $\lim_{\beta \rightarrow \infty} f(\beta) \geq 0$ implying that $f(\beta) > 0$ for all $\beta \geq 0$. \square

By Lemma 2.1.1, it makes sense in the case $\beta_c = \infty$ to define $\lambda'(\beta_c) = \text{ess sup } V$. With this convention, we can prove the following lemma about the minimum of the Hamiltonian taken over the vertices in the n th generation.

Lemma 2.1.3. *We have*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \max_{v \in T_n} \sum_{j=1}^n V(v_j) = \lim_{\beta \rightarrow \infty} \frac{\varphi(\beta)}{\beta} = \lambda'(\beta_c).$$

Proof. Clearly, for $\beta > 0$,

$$\exp \left\{ \beta \max_{v \in T_n} \sum_{j=1}^n V(v_j) \right\} \leq Z_n(\beta) \leq d^n \exp \left\{ \beta \max_{v \in T_n} \sum_{j=1}^n V(v_j) \right\}.$$

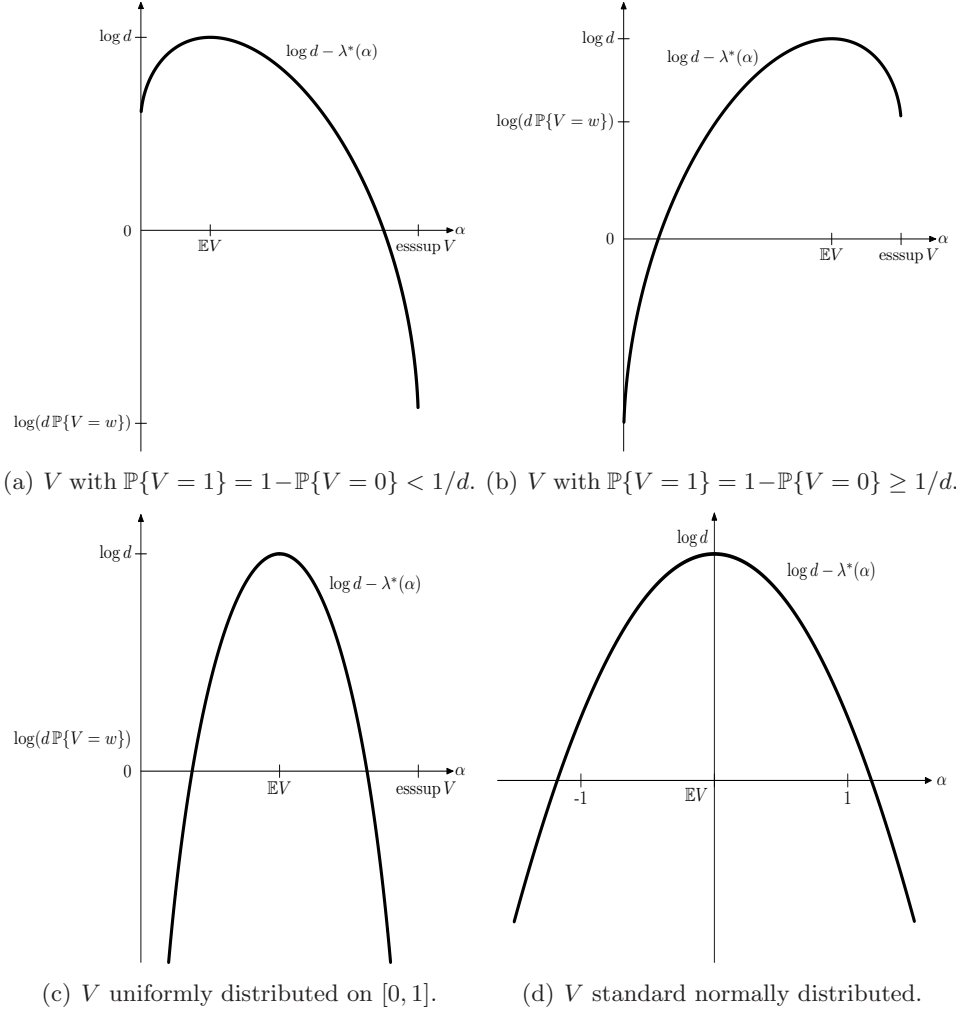


Figure 2-2: The function $\alpha \mapsto \log d - \lambda^*(\alpha)$ for four typical cases. Writing $w = \text{esssup } V$, Figure (a) shows the case that V is bounded, but $0 < \mathbb{P}\{V = w\} < \frac{1}{d}$, whereas in (b) V is bounded, but $\mathbb{P}\{V = w\} \geq \frac{1}{d}$, in (c) V is still bounded, but $\mathbb{P}\{V = w\} = 0$. Finally, in (d), V is unbounded.

Hence, it follows that

$$\frac{1}{n\beta} \log Z_n(\beta) - \frac{1}{\beta} \log d \leq \max_{v \in T_n} \frac{\sum_{j=1}^n V(v_j)}{n} \leq \frac{1}{n\beta} \log Z_n(\beta). \quad (2.2)$$

If $\beta_c < \infty$, then we know from (1.4) that

$$\lim_{\beta \rightarrow \infty} \frac{\varphi(\beta)}{\beta} = \frac{\lambda(\beta_c) + \log d}{\beta_c} = \lambda'(\beta_c),$$

by definition of β_c . Moreover, if $\beta_c = \infty$, then again by (1.4), we know that $\varphi(\beta) = \lambda(\beta) + \log d$ for all $\beta > 0$. Also, the Legendre-Fenchel transform of λ satisfies $\lambda^*(\lambda'(\beta)) = \lambda'(\beta)\beta - \lambda(\beta)$ and the proof of Lemma 2.1.2 shows that $\lim_{\beta \rightarrow \infty} \lambda^*(\lambda'(\beta)) = -\log \mathbb{P}\{V = w\}$. Therefore, we can conclude that

$$\lim_{\beta \rightarrow \infty} \frac{\varphi(\beta)}{\beta} = \lim_{\beta \rightarrow \infty} \frac{\lambda(\beta)}{\beta} = \lim_{\beta \rightarrow \infty} \left(\lambda'(\beta) - \frac{\lambda^*(\lambda'(\beta))}{\beta} \right) = \text{ess sup } V = \lambda'(\beta_c),$$

by Lemma 2.1.1 and the convention $\lambda'(\beta_c) = \text{ess sup } V$. Hence, in either case, letting first $n \rightarrow \infty$ and then $\beta \rightarrow \infty$ in (2.2) yields the statement of Lemma 2.1.3. \square

2.2 Ergodic theory and the multifractal spectrum

In the next two sections we concentrate on the weak disorder phase, in other words we assume that $\beta < \beta_c$, so that the martingale limit $M^{(\beta)}$ is positive.

2.2.1 Ergodic theory on weighted trees

We develop the ergodic theory for a tree with attached weights in analogy to the ergodic theory on Galton-Watson trees developed by Lyons, Pemantle and Peres in [LPP95]. We take advantage of the fact that, in the weak disorder regime, the martingale convergence can be used to construct the infinite-volume Gibbs measure on the boundary of the tree. For this purpose, we extend a finite length polymer $v = (v_0, \dots, v_n)$ to an infinite length polymer $v^+ \in \partial T$ by defining v_{i+1} to be the left-most child of v_i for all $i \geq n$. This enables us to interpret the finite volume Gibbs measures $\mu_n^{(\beta)}$ as probability measures on the boundary ∂T using the convention $\mu_n^{(\beta)}(v^+) = \mu_n^{(\beta)}(v)$ for any $v \in T_n$. We will frequently use this identification in the sequel.

For a vertex $v \in T_n$, let $B(v) = \{\xi \in \partial T : \xi_n = v\}$ and let $T(v)$ be the subtree consisting of all vertices that have v as an ancestor, with v as a root. Then we can define the infinite-volume Gibbs measure $\mu^{(\beta)}$ by

$$\mu^{(\beta)}(B(v)) := e^{\beta \sum_{j=1}^n V(v_j) - n(\lambda(\beta) + \log d)} \frac{M^{(\beta)}(v)}{M^{(\beta)}}$$

where $M^{(\beta)}(v)$ is defined as the almost sure limit of

$$M_n^{(\beta)}(v) = \sum_{w \in T_n(v)} \exp \left(\beta \sum_{j=1}^n V(w_j) - n(\lambda(\beta) + \log d) \right),$$

which exists since $(M_n^{(\beta)}(v), n \geq 0)$ and $(M_n^{(\beta)}, n \geq 0)$ have the same law. Then, we see that almost surely for v such that $|v| = k$, as $n \rightarrow \infty$,

$$\begin{aligned} \mu_n^{(\beta)}(B(v)) &= \frac{1}{Z_n(\beta)} e^{\beta \sum_{j=1}^k V(v_j)} \sum_{w \in T_{n-k}(v)} e^{\beta \sum_{j=1}^{n-k} V(w_j)} \\ &= e^{\beta \sum_{j=1}^k V(v_j) - k(\lambda(\beta) + \log d)} \frac{M_{n-k}^{(\beta)}(v)}{M_n^{(\beta)}} \rightarrow \mu^{(\beta)}(B(v)), \end{aligned}$$

in other words, almost surely, $\mu_n^{(\beta)}$ converges weakly to $\mu^{(\beta)}$.

The central result of this section is the following proposition.

Proposition 2.2.1. *If $\beta < \beta_c$, for \mathbb{P} -almost every disorder and $\mu^{(\beta)}$ -almost every path $\xi \in \partial T$,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n V(\xi_j) = \lambda'(\beta), \quad (2.3)$$

and

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \mu^{(\beta)}(B(\xi_n)) = f(\beta). \quad (2.4)$$

Let **SpinedTrees** = $\{(\mathcal{V}, \xi) : \mathcal{V} = (V(v) : v \in T), \xi \in \partial T\}$ be the space of weights attached to the vertices of the d -ary tree with marked spine, endowed with the product topology. For any vertex $w \in T$ we denote by $\mathcal{V}(w) = (V(v) : v \in T(w))$ the family of weights on the tree $T(w)$. There is a canonical shift

$$\theta : \mathbf{SpinedTrees} \rightarrow \mathbf{SpinedTrees}, \quad \theta(\mathcal{V}, \xi) = (\mathcal{V}(\xi_1), (\xi_1, \xi_2, \dots)).$$

Our aim is to show that θ is a measure-preserving transformation with respect to the measure

$$\nu(d\mathcal{V} d\xi) = \mu_{\mathcal{V}}^{(\beta)}(d\xi) M_{\mathcal{V}}^{(\beta)} \mathbb{P}(d\mathcal{V}),$$

where the subscript \mathcal{V} indicates the dependence of $\mu_{\mathcal{V}}^{(\beta)}$ and $M_{\mathcal{V}}^{(\beta)}$ on the underlying disorder.

Lemma 2.2.2. *The shift θ is ν -preserving.*

Proof. Let A be a Borel set in **SpinedTrees**. Then,

$$\begin{aligned} \nu(\theta^{-1}A) &= \int \mathbb{I}_{\theta^{-1}A}(\mathcal{V}, \xi) \mu_{\mathcal{V}}^{(\beta)}(d\xi) M_{\mathcal{V}}^{(\beta)} \mathbb{P}(d\mathcal{V}) \\ &= \int \sum_{|v|=1} \mathbb{I}_{\{\xi_1=v\}}(\xi) \mathbb{I}_A(\mathcal{V}(v), (v, \xi_2, \xi_3, \dots)) \mu_{\mathcal{V}}^{(\beta)}(d\xi) M_{\mathcal{V}}^{(\beta)} \mathbb{P}(d\mathcal{V}). \end{aligned} \quad (2.5)$$

For any vertex $v = (v_0, \dots, v_n) \in T$ we interpret $\partial T(v)$ as a subset of ∂T by identifying $(v, \xi_1, \xi_2, \dots) \in \partial T(v)$ with $(v_0, \dots, v_n, \xi_1, \xi_2, \xi_3, \dots) \in \partial T$. Hence, for $v \in T$ and $U \subset \partial T(v)$,

$$\mu_{\mathcal{V}(v)}^{(\beta)}(U) = \frac{\mu_{\mathcal{V}}^{(\beta)}(U)}{\mu_{\mathcal{V}}^{(\beta)}(B(v))}.$$

Hence, recalling that $\mu_{\mathcal{V}}^{(\beta)}(B(v)) = e^{\beta V(v) - \lambda(\beta) - \log d} \frac{M_{\mathcal{V}(v)}^{(\beta)}}{M_{\mathcal{V}}^{(\beta)}}$, and using independence of the weights,

$$\begin{aligned} \nu(\theta^{-1}A) &= \int \sum_{|v|=1} \int \mathbb{I}_A(\mathcal{V}(v), (v, \xi_2, \dots)) \mu_{\mathcal{V}}^{(\beta)}(B(v)) \mu_{\mathcal{V}(v)}^{(\beta)}(d(v, \xi_2, \dots)) M_{\mathcal{V}}^{(\beta)} \mathbb{P}(d\mathcal{V}) \\ &= \frac{1}{d} \sum_{|v|=1} \int e^{\beta V(v) - \lambda(\beta)} \int \mathbb{I}_A(\mathcal{V}(v), (v, \xi_2, \dots)) \mu_{\mathcal{V}(v)}^{(\beta)}(d(v, \xi_2, \dots)) M_{\mathcal{V}(v)}^{(\beta)} \mathbb{P}(d\mathcal{V}) \\ &= \mathbb{E}[e^{\beta V - \lambda(\beta)}] \int \mathbb{I}_A(\mathcal{V}, \xi) \mu_{\mathcal{V}}^{(\beta)}(d\xi) M_{\mathcal{V}}^{(\beta)} \mathbb{P}(d\mathcal{V}) = \nu(A). \end{aligned}$$

□

Lemma 2.2.3. *The shift θ is ergodic.*

Proof. By Proposition 16.6 in [LP09], the shift is ergodic with respect to the measure ν if and only if every set A of weights satisfying

$$\sum_{\substack{\mathcal{V}(v) \in A \\ |v|=1}} \mu_{\mathcal{V}}^{(\beta)}(B(v)) = \mathbb{I}_A(\mathcal{V}) \quad \mathbb{P}\text{-almost surely} \quad (2.6)$$

has $\mathbb{P}(A) \in \{0, 1\}$. Therefore, let A be a set satisfying (2.6), then since, almost surely, all balls have positive measure

$$\mathcal{V} \in A \iff \mathcal{V}(v) \in A \text{ for all } v \text{ such that } |v| = 1. \quad (2.7)$$

By iteration, (2.7) implies that A is a tail event with respect to the independent, identically distributed family of weights. Invoking Kolmogorov's zero-one law, we can deduce that $\mathbb{P}(A) = 0$ or 1 , as required. □

Since by the previous two lemmas θ is ν -preserving and ergodic, the pointwise ergodic theorem gives us that for \mathbb{P} -almost every \mathcal{V} and $\mu_{\mathcal{V}}^{(\beta)}$ -almost every $\xi \in \partial T$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n V(\xi_j) = \nu[V(\xi_1)], \quad (2.8)$$

where $\nu[\cdot]$ denotes the expectation with respect to the measure ν . We find

$$\begin{aligned}\nu[V(\xi_1)] &= \int V(\xi_1) \mu_{\mathcal{V}}^{(\beta)}(d\xi) M_{\mathcal{V}}^{(\beta)} \mathbb{P}(d\mathcal{V}) = \int \sum_{|v|=1} V(v) \mu_{\mathcal{V}}^{(\beta)}\{\xi_1 = v\} M_{\mathcal{V}}^{(\beta)} \mathbb{P}(d\mathcal{V}) \\ &= \sum_{|v|=1} \int V(v) e^{\beta V(v) - \lambda(\beta) - \log d} M_{\mathcal{V}(v)}^{(\beta)} \mathbb{P}(d\mathcal{V}) \\ &= \mathbb{E}[V e^{\beta V - \lambda(\beta)}] \mathbb{E}[M_{\mathcal{V}}^{(\beta)}] = \frac{\mathbb{E}[V e^{\beta V}]}{\mathbb{E}[e^{\beta V}]} = \lambda'(\beta),\end{aligned}$$

where we have used independence and the fact that $\mathbb{E}[M_{\mathcal{V}}^{(\beta)}] = 1$. Hence, we have proved the first part of Proposition 2.2.1. Similarly, for the second part, note that

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \mu_{\mathcal{V}}^{(\beta)}(B(\xi_n)) = \log d + \lambda(\beta) - \lim_{n \rightarrow \infty} \beta \frac{1}{n} \sum_{j=1}^n V(\xi_j) - \lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{M_{\mathcal{V}}^{(\beta)}(\xi_n)}{M_{\mathcal{V}}^{(\beta)}}. \quad (2.9)$$

Hence by the first part of Proposition 2.2.1, it suffices to show that the second limit converges to 0. The following lemma from ergodic theory, which can be found for instance in [LPP95, Lemma 6.2], allows us to evaluate the last term.

Lemma 2.2.4. *If S is a measure-preserving transformation on a probability space, g is finite and measurable, and $g - Sg$ is bounded below by an integrable function, then $g - Sg$ is integrable with integral 0.*

Looking at $g(\mathcal{V}, \xi) = \log M_{\mathcal{V}}^{(\beta)}$ and using that $M_{\mathcal{V}(\xi_1)}^{(\beta)} = M_{\mathcal{V}}^{(\beta)}(\xi_1)$, we obtain

$$\begin{aligned}g - \theta g &= \log M_{\mathcal{V}}^{(\beta)} - \log M_{\mathcal{V}}^{(\beta)}(\xi_1) = -\log \mu_{\mathcal{V}}^{(\beta)}(B(\xi_1)) + \beta V(\xi_1) - \lambda(\beta) - \log d \\ &\geq \beta V(\xi_1) - \lambda(\beta) - \log d,\end{aligned}$$

where the latter is integrable. Hence by the ergodic theorem and Lemma 2.2.4, for \mathbb{P} -almost every disorder \mathcal{V} and $\mu_{\mathcal{V}}^{(\beta)}$ -almost every ξ ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{M_{\mathcal{V}}^{(\beta)}(\xi_n)}{M_{\mathcal{V}}^{(\beta)}} = \lim_{n \rightarrow \infty} \frac{-1}{n} \sum_{j=1}^n \log \frac{M_{\mathcal{V}}^{(\beta)}(\xi_{j-1})}{M_{\mathcal{V}}^{(\beta)}(\xi_j)} = -\nu[\log M_{\mathcal{V}}^{(\beta)} - \log M_{\mathcal{V}}^{(\beta)}(\xi_1)] = 0.$$

Therefore (2.9) together with the first part implies the second part of Proposition 2.2.1.

2.2.2 A coarse multifractal spectrum

We use the ergodic theory developed in the previous section to give a direct proof of the following *coarse multifractal spectrum*. As we have seen in Remark 1.1.6, an alternative proof could be obtained by the Gärtner-Ellis theorem.

Proposition 2.2.5. *For all $\alpha \geq \mathbb{E}V$ with $\lambda^*(\alpha) < \log d$, almost surely,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \# \left\{ v \in T_n : \sum_{j=1}^n V(v_j) \geq \alpha n \right\} = \log d - \lambda^*(\alpha).$$

Proof. First note that the proof of Lemma 2.1.2 shows that $\alpha\beta - \lambda(\beta)$ is maximised at $\beta \in [0, \beta_c)$ such that $\alpha = \lambda'(\beta)$. For the *upper bound* consider

$$\begin{aligned} \varphi(\beta) &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{v \in T_n} e^{\beta \sum_{j=1}^n V(v_j)} \geq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{v \in T_n} e^{\beta \sum_{j=1}^n V(v_j)} \mathbb{I}_{\{\sum_{j=1}^n V(v_j) \geq \alpha n\}} \\ &\geq \alpha\beta + \limsup_{n \rightarrow \infty} \frac{1}{n} \log \# \left\{ v \in T_n : \sum_{j=1}^n V(v_j) \geq \alpha n \right\}. \end{aligned}$$

Now, by the expression for the free energy in (1.4), we know that $\varphi(\beta) = \lambda(\beta) + \log d$ for $\beta < \beta_c$. Therefore, rearranging the previous display yields

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \# \left\{ v \in T_n : \sum_{j=1}^n V(v_j) \geq \alpha n \right\} &\leq \varphi(\beta) - \alpha\beta \\ &= \log d + \lambda(\beta) - \lambda'(\beta)\beta = \log d - \lambda^*(\alpha), \end{aligned}$$

where we used the definition of β as the maximizer of the Legendre-Fenchel transform.

For the *lower bound*, recall that λ^* is continuous on its domain and consider $\varepsilon > 0$ small enough such that $\log d - \lambda^*(\alpha + \varepsilon) > 0$. In particular, we can find $0 < \beta < \beta_c$ such that $\alpha + \varepsilon = \lambda'(\beta)$. Then, consider the set

$$E = \left\{ \xi \in \partial T : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n V(\xi_j) = \lambda'(\beta), \lim_{n \rightarrow \infty} -\frac{1}{n} \log \mu^{(\beta)}(B(\xi_n)) = f(\beta) \right\}.$$

By Proposition 2.2.1, $\mu^{(\beta)}(E) = 1$. Moreover, recalling that $\lambda'(\beta) - \varepsilon = \alpha$, for any $k \in \mathbb{N}$, the set E is covered by the collection

$$\bigcup_{n=k}^{\infty} \bigcup_{|v|=n} \{v : \sum_{j=1}^n V(v_j) \geq \alpha n, \mu^{(\beta)}(B(v)) \leq e^{-n(f(\beta)-\varepsilon)}\}.$$

Hence, if we write $q = \liminf_{n \rightarrow \infty} \frac{1}{n} \log \# \{v \in T_n : \sum_{j=1}^n V(v_j) \geq \alpha n\}$, we obtain for k sufficiently large

$$\begin{aligned} 1 &= \mu^{(\beta)}(E) \leq \sum_{n=k}^{\infty} \sum_{|v|=n} \mathbb{I}_{\{\sum_{j=1}^n V(v_j) \geq \alpha n\}} \mathbb{I}_{\{\mu^{(\beta)}(B(v)) \leq e^{-n(f(\beta)-\varepsilon)}\}} \mu^{(\beta)}(B(v)) \\ &\leq \sum_{n=k}^{\infty} \# \{v \in T_n : \sum_{j=1}^n V(v_j) \geq \alpha n\} e^{-n(f(\beta)-\varepsilon)} \leq \sum_{n=k}^{\infty} e^{n(q-f(\beta)+2\varepsilon)}. \end{aligned}$$

Therefore, if $q - f(\beta) + 2\varepsilon < 0$, the sum on the right hand side converges, so by taking k large enough we could make the right hand side < 1 contradicting $\mu^{(\beta)}(E) = 1$. Thus, we conclude that $q - f(\beta) + 2\varepsilon \geq 0$.

Finally, we recall that $f(\beta) = \log d + \lambda(\beta) - \beta\lambda'(\beta) = \log d - \lambda^*(\alpha + \varepsilon)$ so that we have shown that

$$q = \liminf_{n \rightarrow \infty} \frac{1}{n} \log \# \left\{ v \in T_n : \sum_{j=1}^n V(v_j) \geq \alpha n \right\} \geq f(\beta) - 2\varepsilon = \log d - \lambda^*(\alpha + \varepsilon) - 2\varepsilon.$$

Therefore, recalling that λ^* is continuous, we obtain the required lower bound by letting $\varepsilon \downarrow 0$. \square

2.3 Localization in the weak disorder phase

In this section, we prove Theorem 1.1.5 using the theory developed in Section 2.2.

Lemma 2.3.1. *Suppose $\tilde{T} \subset T$ is any subtree satisfying $\mu^{(\beta)}(\partial\tilde{T}) > 0$, then*

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log |\tilde{T}_n| \geq f(\beta).$$

Proof. Using Frostman's lemma, see e.g. Proposition 2.3 in [Fal97], in combination with (2.4) we infer that the Hausdorff dimension of $\partial\tilde{T}$ must be at least $f(\beta)$. The Hausdorff dimension of the boundary of a tree is the logarithm of its branching rate, which is bounded from above by the lower growth rate, see e.g. [LP09]. \square

The next lemma enables us to choose suitable trees for Theorem 1.1.5(a).

Lemma 2.3.2. *Almost surely, for any $\varepsilon > 0$ there exists a subtree $T^{(\varepsilon)} \subset T$ with $\mu^{(\beta)}(\partial T^{(\varepsilon)}) \geq 1 - \varepsilon$, and a sequence $\delta_n \downarrow 0$ such that, for every $\xi \in \partial T^{(\varepsilon)}$ and $n \geq 1$,*

$$\frac{1}{n} \sum_{j=1}^n V(\xi_j) \geq \lambda'(\beta) - \delta_n \quad \text{and} \quad \mu^{(\beta)}(B(\xi_n)) \geq e^{-n(f(\beta) + \delta_n)}.$$

Proof. Since ∂T is a complete separable metric space and $\mu^{(\beta)}$ is a finite measure, we know that $\mu^{(\beta)}$ is regular, see [Sch05]. By Egorov's theorem, see e.g. [Ash00], we can pick a closed subset $A_\varepsilon \subset \partial T$ with the properties that $\mu^{(\beta)}(A_\varepsilon) \geq 1 - \varepsilon$ and the convergence in Proposition 2.2.1 holds uniformly on A_ε . This means that there exists $\delta_n \downarrow 0$ such that the displayed properties in the lemma hold. Now define

$$T^{(\varepsilon)} = \bigcup_{\xi \in A_\varepsilon} \bigcup_{j=0}^{\infty} \xi_j,$$

the set of all vertices on the rays of A_ε with the tree structure inherited from T . It is clear that $T^{(\varepsilon)}$ is a tree and, as A_ε is compact, we have that $\partial T^{(\varepsilon)} = A_\varepsilon$. \square

Proof of Theorem 1.1.5(a). We show that any one of the trees $T^{(\varepsilon)}$, $\varepsilon > 0$, satisfies the requirements of Theorem 1.1.5(a). Indeed, as the balls $B(v)$, $v \in T_n^{(\varepsilon)}$ are disjoint, we infer from Lemma 2.3.2 that there can be at most $\exp(n(f(\beta) + \delta_n))$ vertices in $T_n^{(\varepsilon)}$. Hence,

$$\frac{1}{n} \log |T_n^{(\varepsilon)}| \leq f(\beta) + \delta_n.$$

Recall that $\mu^{(\beta)}(\partial T^{(\varepsilon)}) > 0$. Combining this with Lemma 2.3.1 we obtain that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log |T_n^{(\varepsilon)}| = f(\beta). \quad (2.10)$$

It remains to show that $T^{(\varepsilon)}$ supports the free energy. By (2.10), almost surely, there exists a sequence $\gamma_n \downarrow 0$, such that for all $n \geq 1$,

$$\frac{1}{n} \log |T_n^{(\varepsilon)}| \geq f(\beta) - \gamma_n.$$

Using Lemma 2.3.2 again, we see that

$$\frac{1}{n} \log \left(\sum_{v \in T_n^{(\varepsilon)}} e^{\beta \sum_{j=1}^n V(v_j)} \right) \geq \frac{1}{n} \log \left(e^{n(\lambda'(\beta)\beta - \delta_n)} |T_n^{(\varepsilon)}| \right) \geq \lambda'(\beta)\beta - \delta_n + f(\beta) - \gamma_n,$$

which converges to $\lambda'(\beta)\beta + f(\beta) = \varphi(\beta)$. The opposite bound is trivial, hence the proof of Theorem 1.1.5(a) is complete. \square

The proof of Theorem 1.1.5(a) immediately gives the following corollary.

Corollary 2.3.3. *Almost surely, for every $\beta < \beta_c$ and for every $0 < \varepsilon < 1$, there exists a tree $T^{(\varepsilon)} \subset T$ of growth rate*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log |T_n^{(\varepsilon)}| = f(\beta)$$

such that

$$\mu^{(\beta)}\{\xi \in \partial T^{(\varepsilon)}\} \geq 1 - \varepsilon.$$

We can now proceed with the second part of the proof of Theorem 1.1.5.

Proof of Theorem 1.1.5(b). Since $f(\beta)$ is strictly decreasing on $(0, \beta_c)$, we can choose $\beta < \beta' < \beta_c$ such that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log |A_n| < f(\beta') < f(\beta).$$

Now, choose $\varepsilon > 0$ small enough such that, for all n sufficiently large,

$$|A_n| \leq e^{n(f(\beta') - \varepsilon)}.$$

By Proposition 2.2.5 we have, for large n ,

$$\#\left\{v \in T_n : \sum_{j=1}^n V(v_j) \geq n\lambda'(\beta')\right\} \geq e^{n(f(\beta')-\varepsilon)} \geq |A_n|. \quad (2.11)$$

Next, order the vertices v^1, \dots, v^{d^n} in the n th generation of T such that

$$\sum_{j=1}^n V(v_j^1) \geq \sum_{j=1}^n V(v_j^2) \geq \dots \geq \sum_{j=1}^n V(v_j^{d^n}).$$

Then, clearly

$$\begin{aligned} \sum_{v \in A_n} e^{\beta \sum_{j=1}^n V(v_j)} &\leq \sum_{k=1}^{|A_n|} e^{\beta \sum_{j=1}^n V(v_j^k)} \\ &\leq \sum_{v \in T_n} \mathbb{I}\{v \in T_n : \sum_{j=1}^n V(v_j) \geq n\lambda'(\beta')\} e^{\beta \sum_{j=1}^n V(v_j)}, \end{aligned}$$

where the last inequality follows from (2.11). Note that by Lemma 2.1.3, for large n ,

$$\max_{v \in T_n} \frac{1}{n} \sum_{j=1}^n V(v_j) \leq \lambda'(\beta_c) + \varepsilon.$$

Hence, we can write

$$\begin{aligned} \sum_{v \in T_n} \mathbb{I}\left\{v \in T_n : \sum_{j=1}^n V(v_j) \geq n\lambda'(\beta')\right\} e^{\beta \sum_{j=1}^n V(v_j)} \\ \leq \sum_{i=1}^N \#\left\{v \in T_n : \alpha_{i-1} \leq \frac{1}{n} \sum_{j=1}^n V(v_j) \leq \alpha_i + \varepsilon\right\} e^{\beta n(\alpha_i + \varepsilon)}, \end{aligned}$$

where $\alpha_i = (1 - \frac{i}{N})\lambda'(\beta') + \frac{i}{N}\lambda'(\beta_c)$, for $i = 1, \dots, N$ and some fixed N . Writing $\varphi^*(\alpha) = \sup_{\tau \in \mathbb{R}} \{\alpha\tau - \varphi(\tau)\}$ for the Legendre-Fenchel transform of φ , we find that by Proposition 2.2.5 again, for n sufficiently large,

$$\#\{v \in T_n : \sum_{j=1}^n V(v_j) \geq n\alpha_{i-1}\} \leq e^{n(-\varphi^*(\alpha_{i-1})+\varepsilon)}.$$

Combining the previous displays and taking $N > \frac{1}{\varepsilon}$ such that $\alpha_i \leq \alpha_{i-1} + \varepsilon$, we obtain

$$\begin{aligned} \sum_{v \in A_n} e^{\beta \sum_{j=1}^n V(v_j)} &\leq \sum_{i=1}^N e^{n(\beta\alpha_{i-1} - \varphi^*(\alpha_{i-1}) + (1+2\beta)\varepsilon)} \\ &\leq N \exp \left\{ n \left(\max_{\alpha \in [\lambda'(\beta'), \lambda'(\beta_c)]} (\beta\alpha - \varphi^*(\alpha)) + (1+2\beta)\varepsilon \right) \right\}. \end{aligned} \quad (2.12)$$

Since $\lambda'(\beta') > \mathbb{E}V$ it follows that

$$\max_{\alpha \in [\lambda'(\beta'), \lambda'(\beta_c)]} (\beta\alpha - \varphi^*(\alpha)) \leq \max_{\alpha \in [\mathbb{E}V, \lambda'(\beta_c)]} (\beta\alpha - \varphi^*(\alpha)) = \varphi(\beta), \quad (2.13)$$

where the last equality follows from the Legendre-Fenchel duality. But now, by (1.4), $\varphi = \lambda + \log d$ on the set $[0, \beta_c]$ and therefore φ is differentiable with derivative λ' . Legendre-Fenchel duality implies that φ^* is strictly convex on $[\mathbb{E}V, \lambda'(\beta_c)]$. In particular, since the maximum on the right hand side in (2.13) is achieved at $\lambda'(\beta)$ and $\lambda'(\beta') > \lambda'(\beta)$, it follows that the inequality is in fact strict. Hence, we can additionally assume that ε is small enough such that

$$\max_{\alpha \in [\lambda'(\beta'), \lambda'(\beta_c)]} (\beta\alpha - \varphi^*(\alpha)) + 2(1 + \beta)\varepsilon < \varphi(\beta).$$

Then, for n large enough such that $\frac{1}{n} \log N < \varepsilon$, we can combine the previous display with (2.12) to obtain the required inequality,

$$\frac{1}{n} \log \sum_{v \in A_n} e^{\beta \sum_{j=1}^n V(v_j)} \leq \frac{1}{n} \log N + \max_{\alpha \in [\lambda'(\beta'), \lambda'(\beta_c)]} (\beta\alpha - \varphi^*(\alpha)) + (1 + 2\beta)\varepsilon < \varphi(\beta),$$

which completes the proof of Theorem 1.1.5(b). \square

2.4 Localization in the critical regime

In this section, we prove Theorem 1.1.7, in other words, we show that in the critical and supercritical case a single ray supports the free energy. Recalling our convention that $\lambda'(\beta_c) = \text{ess sup } V$, if $\beta_c = \infty$, Theorem 1.1.7 follows immediately from the following proposition. Although the result looks similar to (2.3), its proof is considerably more involved as it deals with the critical case.

Proposition 2.4.1. *Almost surely, there exists a ray $\xi \in \partial T$ such that*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n V(\xi_j) = \lambda'(\beta_c).$$

The proofs in this section use ideas from branching random walks as developed in Biggins and Kyprianou [BK04] and Hambly et al. [HKK03]. We split the proof in two parts according to whether β_c is finite or infinite.

2.4.1 Proof of Proposition 2.4.1 when $\beta_c < \infty$

Suppose that f has a positive root, i.e. $\beta_c < \infty$. Recall that in this case $\lambda(\beta_c) + \log d = \beta_c \lambda'(\beta_c)$, which we will use frequently throughout this section. The idea of the proof is to restrict attention to those polymers where the average of the weights is smaller than

the critical weight $\lambda'(\beta_c)$. More precisely, introduce the cemetery state Δ and define new weights by setting for $v \in T_n$ and for $x \geq 0$,

$$\tilde{V}^x(v) = \begin{cases} V(v) & \text{if } \sum_{j=1}^k V(v_j) < x + k\lambda'(\beta_c) \text{ for all } k \leq n, \\ \Delta & \text{otherwise.} \end{cases}$$

Therefore, it is clear that if the weight associated to v is Δ , then all the descendants of v also have the weight Δ . Moreover, for $x = 0$ we omit the superscript and write $\tilde{V} := \tilde{V}^0$. This construction corresponds to a killed branching random walk.

The aim is now to define a martingale which induces a change of measure such that under the new measure there exists a ray with critical weight. First of all, introduce a size-biased version V^* of V , whose distribution is given by

$$\mathbb{E}[g(V^*)] = \mathbb{E}[g(V)e^{\beta_c V - \lambda(\beta_c)}],$$

for any bounded, measurable function g . Note that

$$\mathbb{E}[V^*] = \frac{\mathbb{E}[Ve^{\beta_c V}]}{\mathbb{E}[e^{\beta_c V}]} = \lambda'(\beta_c).$$

Therefore, if $(V_j^*, j \geq 1)$ is a sequence of independent random variables with the same distribution as V^* , then the random walk with increments given by (V_j^*) has a drift $\lambda'(\beta_c)$. Now, define $\tau = \inf\{n \geq 1 : \sum_{j=1}^n V_j^* < n\lambda'(\beta_c)\}$ as the first time that the random walk with increments (V_j^*) grows slower than its drift. Then, we set for $x > 0$,

$$h(x) = \mathbb{E}\left[\sum_{n=0}^{\tau} \mathbb{I}\left\{\sum_{j=1}^n V_j^* - n\lambda'(\beta_c) \in [0, x]\right\}\right],$$

as the expected number of visits of the normalised random walk with increments $(V_j^* - \lambda'(\beta_c))$ to $[0, x)$ before hitting $(-\infty, 0)$. Furthermore, we set $h(0) = 1$.

For $x \geq 0$, we define the martingale $(W_n^x : n \geq 0)$ by

$$W_n^x = \sum_{v \in T_n} \frac{h(x - \sum_{j=1}^n V(v_j) + n\lambda'(\beta_c))}{h(x)} e^{\beta_c \sum_{j=1}^n V(v_j) - n(\lambda(\beta_c) + \log d)} \mathbb{I}_{\{\tilde{V}^x(v) \neq \Delta\}}.$$

Again, for $x = 0$ we omit the superscript and write $W_n = W_n^0$. In order to prove that this defines a martingale, we need the following facts, see Lemma 10.1 in [BK04].

Lemma 2.4.2.

- (i) As $x \rightarrow \infty$, $\frac{h(x)}{x} \rightarrow C$, for some constant $C > 0$.
- (ii) For $x \geq 0$, we have $\mathbb{E}[h(x - V^* + \lambda'(\beta_c)) \mathbb{I}\{x - V^* + \lambda'(\beta_c) > 0\}] = h(x)$.

Now, the proof that $(W_n^x : n \geq 0)$ is a martingale with respect to the filtration given by $\mathcal{F}_n = \sigma(V(v) : |v| \leq n)$ is a straight-forward calculation.

Lemma 2.4.3. *The process $(W_n^x: n \geq 1)$ defines a martingale of mean one.*

Proof. Recall that $\lambda(\beta_c) + \log d = \beta_c \lambda'(\beta_c)$. Then

$$\begin{aligned}
& h(x) \mathbb{E}[W_{n+1}^x \mid \mathcal{F}_n] \\
&= \mathbb{E} \left[\sum_{v \in T_{n+1}} h \left(x - \sum_{j=1}^{n+1} (V(v_j) - \lambda'(\beta_c)) \right) e^{\beta_c \sum_{j=1}^{n+1} (V(v_j) - \lambda'(\beta_c))} \mathbb{I}\{\tilde{V}^x(v) \neq \Delta\} \mid \mathcal{F}_n \right] \\
&= \sum_{\substack{v \in T_n \\ \tilde{V}^x(v) \neq \Delta}} \sum_{w \in T_1(v)} \mathbb{E} \left[h \left(x - \sum_{j=1}^n V(v_j) - V(w) + (n+1)\lambda'(\beta_c) \right) \right. \\
&\quad \times e^{\beta_c (\sum_{j=1}^n V(v_j) + V(w) - (n+1)\lambda'(\beta_c))} \mathbb{I}\left\{ \sum_{j=1}^n V(v_j) + V(w) < x + (n+1)\lambda'(\beta_c) \right\} \mid \mathcal{F}_n \Big].
\end{aligned}$$

Now note that $V(w)$ is independent of \mathcal{F}_n and recall the definition of V^* . Then we can continue the display with

$$\begin{aligned}
&= \sum_{\substack{v \in T_n \\ \tilde{V}^x(v) \neq \Delta}} e^{\beta_c (\sum_{j=1}^n V(v_j) - n\lambda'(\beta_c))} \mathbb{E} \left[h \left(x - \sum_{j=1}^n V(v_j) + n\lambda'(\beta_c) - V^* + \lambda'(\beta_c) \right) \right. \\
&\quad \times \mathbb{I}\left\{ x - \sum_{j=1}^n V(v_j) + n\lambda'(\beta_c) - V^* + \lambda'(\beta_c) > 0 \right\} \Big] \\
&= \sum_{\substack{v \in T_n \\ \tilde{V}^x(v) \neq \Delta}} h \left(x - \sum_{j=1}^n V(v_j) + n\lambda'(\beta_c) \right) e^{\beta_c (\sum_{j=1}^n V(v_j) - n\lambda'(\beta_c))} = h(x) W_n^x,
\end{aligned}$$

where we have used Lemma 2.4.2 (ii). This lemma also confirms that W_n^x has mean 1. \square

Allowing the cemetery state as a possible weight in **SpinedTrees** we can, similarly as in Section 2.2.1, extend the measure \mathbb{P} to a measure \mathbb{P}^* on **SpinedTrees** by choosing the spine uniformly, i.e. by choosing ξ_{n+1} with equal probability from the children of ξ_n . Define the extended filtration

$$\mathcal{F}_n^* = \sigma(\mathcal{F}_n, \xi_i, i = 1, \dots, n).$$

We now perform a change of measure such that the weights $(V(\xi_i))$ along the spine will be chosen such that $\sum_{j=1}^n (V(\xi_j) - \lambda'(\beta_c))$ follows the law of a random walk conditioned to stay negative. More precisely, define the probability measure \mathbb{Q}^* via

$$\left. \frac{d\mathbb{Q}^*}{d\mathbb{P}^*} \right|_{\mathcal{F}_n^*} = h \left(n\lambda'(\beta_c) - \sum_{j=1}^n V(\xi_j) \right) e^{\beta_c \sum_{j=1}^n V(\xi_j) - n\lambda(\beta)} \mathbb{I}\{\tilde{V}(\xi_n) \neq \Delta\}.$$

This construction defines a probability measure, since the left-hand side is a martingale under \mathbb{P}^* .

From the definition it follows that under the new measure \mathbb{Q}^* , the distribution of the weights is constructed as follows:

- The spine ξ is chosen uniformly, i.e. ξ_{n+1} is chosen uniformly among the children of ξ_n .
- The weights along the spine ξ are distributed such that their average is conditioned to be less than the critical weight $\lambda'(\beta_c)$, i.e. if at time n the weights along the spine satisfy $s = \sum_{j=1}^n V(\xi_j) < n\lambda'(\beta_c)$, then the weight for ξ_{n+1} is chosen according to Doob's h -transform,

$$\begin{aligned} \mathbb{Q}^* \left[V(\xi_{n+1}) \in dz \mid \sum_{j=1}^n V(\xi_j) = s \right] \\ = \frac{h((n+1)\lambda'(\beta_c) - (z+s))}{h(n\lambda'(\beta_c) - s)} \mathbb{I}\{z+s < (n+1)\lambda'(\beta_c)\} e^{\beta_c z - \lambda(\beta_c)} \mathbb{P}\{V \in dz\}. \end{aligned}$$

- The weights of the vertices not on the spine remain unaffected by the change of measure. In other words, if η_n is a sibling of ξ_n , then we generate a weight $V(\eta_n)$ with the distribution of V and attach it to η_n if

$$\sum_{j=1}^{n-1} V(\xi_j) + V(\eta_n) < n\lambda'(\beta_c),$$

and otherwise η_n receives the weight Δ . Then conditionally on $\tilde{V}(\eta_n) \neq \Delta$, the random disorder in the tree started in η_n is given by the weights $(\tilde{V}^x(v) : v \in T(\eta_n))$ for

$$x = n\lambda'(\beta_c) - \sum_{j=1}^{n-1} V(\xi_j) - V(\eta_n).$$

If we restrict \mathbb{Q}^* to the σ -algebra $\mathcal{F} = \sigma(\bigcup_{n \geq 1} \mathcal{F}_n)$, we obtain a measure \mathbb{Q} defined on the space of trees with weights. Moreover, we obtain its density on \mathcal{F}_n .

Lemma 2.4.4.

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_n} = W_n.$$

Proof. Writing $\mathbb{P}^*[\cdot]$ for the expectation with respect to \mathbb{P}^* , we obtain from the definition of conditional expectation

$$\begin{aligned} \frac{d\mathbb{Q}^*}{d\mathbb{P}^*} \Big|_{\mathcal{F}_n} &= \mathbb{P}^* \left[h \left(n\lambda'(\beta_c) - \sum_{j=1}^n V(\xi_j) \right) e^{\beta_c \sum_{j=1}^n V(\xi_j) - n\lambda(\beta)} \mathbb{I}\{\tilde{V}(\xi_n) \neq \Delta\} \mid \mathcal{F}_n \right] \\ &= \mathbb{P}^* \left[\sum_{v \in T_n} \mathbb{I}_{\{\xi_n=v\}} h \left(n\lambda'(\beta_c) - \sum_{j=1}^n V(v_j) \right) e^{\beta_c \sum_{j=1}^n V(v_j) - n\lambda(\beta)} \mathbb{I}_{\{\tilde{V}(v) \neq \Delta\}} \mid \mathcal{F}_n \right] \end{aligned}$$

$$= \sum_{\substack{v \in T_n \\ \tilde{V}(v) \neq \Delta}} h\left(n\lambda'(\beta_c) - \sum_{j=1}^n V(v_j)\right) e^{\beta_c \sum_{j=1}^n V(v_j) - n\lambda(\beta)} \mathbb{P}^*\{\xi_n = v\} = W_n,$$

which proves the claim. \square

The next step will be to show that \mathbb{Q} is absolutely continuous with respect to \mathbb{P} .

Lemma 2.4.5. *\mathbb{Q} is absolutely continuous with respect to \mathbb{P} with Radon-Nikodým derivative*

$$W := \limsup_{n \rightarrow \infty} W_n.$$

Furthermore, \mathbb{Q}^* -almost surely

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n V(\xi_j) = \lambda'(\beta_c).$$

Proof. By a standard measure theoretic result, see for instance Lemma 12.2 in [LP09],

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = W \iff W < \infty \quad \mathbb{Q}\text{-almost surely.} \quad (2.14)$$

Denote by $\mathcal{G} = \sigma(V(\xi_k) : k = 1, 2, \dots)$ the σ -algebra containing all the information about the weights along the spine. The first step is to calculate the conditional expectation $\mathbb{Q}^*[W_n | \mathcal{G}]$. With this in mind, consider a path $v \in T_n$. Decomposing according to the last common ancestor with the spine,

$$\begin{aligned} & \mathbb{Q}^* \left[h\left(n\lambda'(\beta_c) - \sum_{j=1}^n V(v_j)\right) e^{\beta_c \sum_{j=1}^n V(v_j) - n(\lambda(\beta_c) + \log d)} \mathbb{I}\{\tilde{V}(v) \neq \Delta\} \middle| \mathcal{G} \right] \\ &= \sum_{m=0}^n h\left(m\lambda'(\beta_c) - \sum_{j=1}^m V(\xi_j)\right) e^{\beta_c \sum_{j=1}^m V(\xi_j) - m(\lambda(\beta_c) + \log d)} \\ & \quad \times \mathbb{Q}^*\{\max\{k : v_k = \xi_k\} = m\} \\ & \quad \times \mathbb{E} \left[\prod_{i=m+1}^n \frac{h(i\lambda'(\beta_c) - \sum_{j=1}^i V(v_j))}{h((i-1)\lambda'(\beta_c) - \sum_{j=1}^{i-1} V(v_j))} e^{\beta_c V(v_i) - \lambda(\beta_c)} \mathbb{I}_{\{\sum_{j=1}^i V(v_j) < i\lambda'(\beta_c)\}} \middle| \mathcal{F}_m \right] \\ &\leq \sum_{m=0}^n h\left(m\lambda'(\beta_c) - \sum_{j=1}^m V(\xi_j)\right) d^{-n} e^{\beta_c \sum_{j=1}^m V(\xi_j) - m(\lambda(\beta_c) + \log d)}, \end{aligned}$$

where we used the fact that under \mathbb{Q}^* the weights of the vertices not on the spine have the same distribution as under \mathbb{P}^* , so that we can apply Lemma 2.4.2 (ii) repeatedly to show that the conditional expectation of the product is equal to 1. Summing over all $v \in T_n$ we obtain from the previous equation

$$\mathbb{Q}^*[W_n | \mathcal{G}] \leq \sum_{m=0}^n h\left(m\lambda'(\beta_c) - \sum_{j=1}^m V(\xi_j)\right) e^{\beta_c \sum_{j=1}^m V(\xi_j) - m(\lambda(\beta_c) + \log d)}.$$

Recall that $\sum_{j=1}^n V(\xi_j)$ under \mathbb{Q}^* has the law of a random walk conditioned to stay strictly below $n\lambda'(\beta_c)$. In other words, $-\sum_{j=1}^n V(\xi_j) + n\lambda'(\beta_c)$ follows the law of a random walk conditioned to stay positive. It is known (see, for instance, [HKK03] where the case of a random walk conditioned to stay non-negative was treated) that \mathbb{Q}^* -almost surely for any $\varepsilon > 0$, there exist constants $C_1, C_2 > 0$ such that for all sufficiently large n ,

$$C_1 n^{\frac{1}{2}-\varepsilon} \leq -\sum_{j=1}^n V(\xi_j) + n\lambda'(\beta_c) \leq C_2 n^{\frac{1}{2}+\varepsilon}. \quad (2.15)$$

Hence, using that by Lemma 2.4.2, $h(x)/x \rightarrow C$ as $x \rightarrow \infty$, the previous estimate shows that, \mathbb{Q}^* -almost surely

$$\limsup_{n \rightarrow \infty} \mathbb{Q}^*[W_n | \mathcal{G}] < \infty.$$

By Fatou's lemma we can conclude that $\liminf_{n \rightarrow \infty} W_n$ is also \mathbb{Q}^* -almost surely finite, so in particular it is \mathbb{Q} -almost surely finite. From the representation in Lemma 2.4.4, we see that $1/W_n$ is a nonnegative super-martingale under \mathbb{Q} and hence it has a \mathbb{Q} -almost sure limit. Hence, \mathbb{Q} -almost surely $W = \limsup_{n \rightarrow \infty} W_n = \liminf_{n \rightarrow \infty} W_n < \infty$, so that by (2.14), \mathbb{Q} is absolutely continuous with respect to \mathbb{P} with Radon-Nikodým derivative W . Moreover, (2.15) shows that \mathbb{Q}^* -almost surely

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n V(\xi_j) = \lambda'(\beta_c).$$

□

Now, we are finally in the position to complete the proof of Proposition 2.4.1.

Proof of Proposition 2.4.1 when $\beta_c < \infty$. By Lemma 2.4.5, we know that \mathbb{Q}^* -almost surely, the weights along the spine satisfy

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n V(\xi_j) = \lambda'(\beta_c). \quad (2.16)$$

Now projecting down onto \mathcal{F} , we see that \mathbb{Q} -almost surely there exists a ray $\xi \in \partial T$ that satisfies (2.16). But since \mathbb{Q} is absolutely continuous with respect to \mathbb{P} , we can deduce that

$$\mathbb{P}\left\{ \text{there exists } \xi \in \partial T \text{ with } \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n V(\xi_j) = \lambda'(\beta_c) \right\} > 0.$$

But the event in question is a tail event with respect to the independent, identically distributed family of weights, so that by Kolmogorov's zero-one law it follows that the event has probability 1. □

2.4.2 Proof of Proposition 2.4.1 when $\beta_c = \infty$

We now consider the case that f does not have a positive root. By Lemma 2.1.2 this implies that $w = \text{ess sup } V$ is finite and $\mathbb{P}\{V = w\} \geq \frac{1}{d}$. We start by considering the special case of a Bernoulli disorder. Therefore, assume that $\mathbb{P}\{V = 1\} = p = 1 - \mathbb{P}\{V = 0\}$ with $p \geq \frac{1}{d}$. At the end of this section we will see that it is easy to generalize the result and to prove Proposition 2.4.1 for a general disorder with $\beta_c = \infty$.

Lemma 2.4.6. *For the Bernoulli disorder with success probability $p \geq \frac{1}{d}$, almost surely, there exists a ray $\xi \in \partial T$ such that*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n V(\xi_j) = 1.$$

As in the previous Section 2.4.1 we use a change of measure argument. In this case, our aim is to produce a new measure under which the spine has an asymptotic average weight equal to 1.

Proof. Fix $p \in [\frac{1}{d}, 1)$. Define a sequence $(p_i)_{i \geq 1}$ of increasing numbers in $[p, 1)$ that converges to 1 by setting $p_i = \max\{(\frac{1}{i})^{2/i}, p\}$. As before, let \mathbb{P} be the probability measure such that the random variables $(V(v) : v \in T)$ are independent random variables with Bernoulli distribution with success probability p . Next, we extend \mathbb{P} to a probability measure \mathbb{P}^* on the set of spined trees such that the spine is chosen uniformly. Also, set $\mathcal{F}_n^* = \sigma(V(v), |v| \leq n, \xi(j), j \leq n)$ and denote its projection onto the trees with random weights by $\mathcal{F}_n = \sigma(V(v), |v| \leq n)$. Then, we can define a new probability measure \mathbb{Q}^* on the set of spined trees by setting

$$\left. \frac{d\mathbb{Q}^*}{d\mathbb{P}^*} \right|_{\mathcal{F}_n^*} = \prod_{i=1}^n \left(\frac{p_i}{p} \right)^{V(\xi_i)} \left(\frac{1-p_i}{1-p} \right)^{1-V(\xi_i)}.$$

It is easy to check that the right hand side defines a martingale under \mathbb{P}^* , which implies that the measure \mathbb{Q}^* is well-defined. Moreover, under the new measure the spine ξ is still chosen uniformly, but $V(\xi_i)$ is now Bernoulli with success probability p_i , whereas if $v \neq \xi_i$, for any i , $V(v)$ is still Bernoulli with success probability p .

Now, we can define \mathbb{Q} as the projection of \mathbb{Q}^* onto $\mathcal{F} = \sigma(\bigcup_{n \geq 1} \mathcal{F}_n)$. Then, as before

$$\begin{aligned} \left. \frac{d\mathbb{Q}^*}{d\mathbb{P}^*} \right|_{\mathcal{F}_n} &= \mathbb{P}^* \left[\prod_{i=1}^n \left(\frac{p_i}{p} \right)^{V(\xi_i)} \left(\frac{1-p_i}{1-p} \right)^{1-V(\xi_i)} \middle| \mathcal{F}_n \right] \\ &= \mathbb{P}^* \left[\sum_{v \in T_n} \mathbb{I}_{\{v=\xi_n\}} \prod_{i=1}^n \left(\frac{p_i}{p} \right)^{V(v)} \left(\frac{1-p_i}{1-p} \right)^{1-V(v)} \middle| \mathcal{F}_n \right] \\ &= \sum_{v \in T_n} \frac{1}{d^n} \prod_{i=1}^n \left(\frac{p_i}{p} \right)^{V(v)} \left(\frac{1-p_i}{1-p} \right)^{1-V(v)} =: M_n. \end{aligned}$$

Clearly, $(M_n, n \geq 0)$ defines a martingale with respect to \mathbb{P} and the filtration $(\mathcal{F}_n, n \geq 0)$. As in the proof of Lemma 2.4.5, our aim will be to show that $M = \limsup_{n \rightarrow \infty} M_n < \infty$, \mathbb{Q} -almost surely. For this purpose define $\mathcal{G} = \sigma(V(\xi_i) : i \geq 1)$ and consider the conditional expectation

$$\begin{aligned} \mathbb{Q}^*[M_n | \mathcal{G}] &= \mathbb{Q}^* \left[\sum_{v \in T_n} \sum_{m=0}^n \mathbb{I}_{\{\max\{k: v_k = \xi_k\} = m\}} \frac{1}{d^n} \prod_{i=1}^n \left(\frac{p_i}{p} \right)^{V(v)} \left(\frac{1-p_i}{1-p} \right)^{1-V(v)} \middle| \mathcal{G} \right] \\ &= \sum_{m=0}^n \prod_{i=1}^m \left(\frac{p_i}{p} \right)^{V(\xi_i)} \left(\frac{1-p_i}{1-p} \right)^{1-V(\xi_i)} \frac{1}{d^n} \#\{v \in T_n : \max\{k : v_k = \xi_k\} = m\} \\ &\leq \sum_{m=0}^n \frac{1}{d^m} \prod_{i=1}^m \left(\frac{p_i}{p} \right)^{V(\xi_i)} \left(\frac{1-p_i}{1-p} \right)^{1-V(\xi_i)}. \end{aligned}$$

Now, recall that $p_i \geq p$ so that $\frac{1-p_i}{1-p} \leq \frac{p_i}{p}$. Hence, using that p_i is increasing and $p \geq \frac{1}{d}$, we can deduce from the previous display that

$$\mathbb{Q}^*[M_n | \mathcal{G}] \leq \sum_{m=0}^n \frac{1}{d^m} p^{-m} \prod_{i=1}^m p_i \leq \sum_{m=0}^n p_m^m.$$

Hence, $\limsup_{n \rightarrow \infty} \mathbb{Q}^*[M_n | \mathcal{G}] < \infty$, since $p_m^m = \frac{1}{m^2}$ for all m large enough. Precisely, as in Section 2.4.1 we can thus deduce by Fatou's lemma that $\liminf_{n \rightarrow \infty} M_n$ is \mathbb{Q}^* -almost surely finite and thus \mathbb{Q} -almost surely finite. By construction, $\frac{1}{M_n}$ is a positive \mathbb{Q} -martingale, which implies that its limit exists and hence $M = \lim_{n \rightarrow \infty} M_n < \infty$, \mathbb{Q} -almost surely. Therefore, \mathbb{Q} is absolutely continuous with respect to \mathbb{P} with Radon-Nikodým derivative M .

We have seen that $\mathbb{Q}^*[V(\xi_i)] = p_i$. Since $p_i \rightarrow 1$ as $i \rightarrow \infty$, it is clear that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{Q}^* \left[\sum_{j=1}^n V(\xi_j) \right] = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n p_j = 1.$$

Now $0 \leq V(\xi_i) \leq 1$ so that by Lebesgue's dominated convergence theorem,

$$\mathbb{Q}^* \left[\limsup_{n \rightarrow \infty} \left(1 - \frac{1}{n} \sum_{j=1}^n V(\xi_j) \right) \right] \leq 1 - \lim_{n \rightarrow \infty} \mathbb{Q}^* \left[\frac{1}{n} \sum_{j=1}^n V(\xi_j) \right] = 0.$$

Since $0 \leq 1 - \frac{1}{n} \sum_{j=1}^n V(\xi_j) \leq 1$, we deduce that \mathbb{Q}^* -almost surely $\frac{1}{n} \sum_{j=1}^n V(\xi_j) \rightarrow 1$ as $n \rightarrow \infty$.

Hence, \mathbb{Q} -almost surely, there exists a ray $\xi \in \partial T$ such that $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n V(\xi_j) = 1$. As \mathbb{Q} is absolutely continuous with respect to \mathbb{P} it follows that

$$\mathbb{P} \left\{ \text{there exists } \xi \in \partial T \text{ with } \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n V(\xi_j) = 1 \right\} > 0.$$

As in the previous section, we deduce from Kolmogorov's zero-one law that this probability is in fact equal to 1, so that we have proved Lemma 2.4.6. \square

We now use the previous lemma for the Bernoulli disorder to complete the proof of Proposition 2.4.1. Assume that V is any random variable such that the corresponding function f has no positive root. Recall that this means that $\mathbb{P}\{V = w\} \geq \frac{1}{d}$ for $w = \text{ess sup } V < \infty$.

Proof of Proposition 2.4.1 when $\beta_c = \infty$. Given the disorder $(V(v), v \in T)$, define the random variables $\tilde{V}(v) = \mathbb{I}\{V(v) = w\}$. Then $p := \mathbb{P}\{\tilde{V}(v) = 1\} = \mathbb{P}\{V(v) = w\} \geq \frac{1}{d}$. Lemma 2.4.6 shows that there exists a ray $\xi \in \partial T$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \tilde{V}(\xi_j) = 1.$$

Therefore,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n V(\xi_j) \geq \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n V(\xi_j) \mathbb{I}\{V(\xi_j) = w\} \geq w \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \tilde{V}(\xi_j) = w.$$

Finally, as the reversed inequality is trivial, we have completed the proof. \square

2.5 ϱ -percolation on regular trees

We now show how the directed polymer model on trees can be interpreted in the framework of ϱ -percolation. Consider a d -ary tree T as before and, for $p \in [0, 1]$, define the disorder $\mathcal{V}_p = (V_p(v) : v \in T)$ as a family of independent, identically distributed Bernoulli random variables with success parameter p . An edge leading to a vertex v with weight $V_p(v) = 1$ is considered to be open and if $V_p(v) = 0$ it is defined to be closed. For $\varrho \in [p, 1]$, we say that ϱ -percolation occurs if there exists a path $\xi \in \partial T$ such that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n V_p(\xi_j) \geq \varrho.$$

Lemma 2.5.1. *Fix $p \in (0, 1)$ and let $\lambda_p(\beta) = \log \mathbb{E}[e^{\beta V_p}]$. Let $\alpha_c(p) = 1$, if $p \geq \frac{1}{d}$, and otherwise let $\alpha_c(p)$ be the unique solution of $\lambda_p^*(\alpha) = \log d$ in the interval $(p, 1)$. Then, if $\alpha \leq \alpha_c(p)$, almost surely, there exists $\xi \in \partial T$ such that*

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n V_p(\xi_j) \geq \alpha,$$

but if $\alpha > \alpha_c(p)$ almost surely no such $\xi \in \partial T$ exists.

Proof. Using Lemma 2.1.2 we see that the critical parameter $\beta_c = \beta_c(p)$ for the polymer model with disorder \mathcal{V}_p is infinite if $p \geq \frac{1}{d}$ and finite otherwise. In the latter case, this

implies that $\alpha_c(p)$ is well-defined and $\alpha_c(p) = \lambda'_p(\beta_c)$. From Proposition 2.4.1 we hence obtain in both cases that there exists a ray $\xi \in \partial T$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n V_p(\xi_j) = \alpha_c(p).$$

To show that there is no ray $\xi \in \partial T$ along which we obtain a larger liminf, we may assume that $p < \frac{1}{d}$. Recall that, by (1.4), the free energy

$$\varphi_p(\beta) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{v \in T_n} e^{\beta \sum_{j=1}^n V_p(v_j)}$$

satisfies $\varphi_p(\beta_c) = \beta_c \alpha_c(p)$. Hence, for any ray $\xi \in \partial T$,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n V_p(\xi_j) \leq \frac{1}{\beta_c} \liminf_{n \rightarrow \infty} \frac{1}{n} \log \sum_{v \in T_n} e^{\beta_c \sum_{j=1}^n V_p(v_j)} = \frac{\varphi_p(\beta_c)}{\beta_c} = \alpha_c(p),$$

which proves the second part of Lemma 2.5.1. \square

As the next step, we give an explicit formula for $\alpha_c(p)$ when $p < \frac{1}{d}$. First, we compute the logarithmic moment generating function and its derivative

$$\lambda_p(\beta) = \log \mathbb{E}[e^{\beta V_p}] = \log(pe^\beta + (1-p)) \quad \text{and} \quad \lambda'_p(\beta) = \frac{pe^\beta}{pe^\beta + (1-p)}.$$

Then, using that $\alpha_c(p) = \lambda'_p(\beta_c(p))$ for the polymer with disorder \mathcal{V}_p , we get

$$\alpha_c(p) = \frac{pe^{\beta_c(p)}}{pe^{\beta_c(p)} + (1-p)}. \quad (2.17)$$

As $\log d = \lambda^*(\alpha_c(p)) = \alpha_c(p)\beta_c(p) - \log(pe^{\beta_c(p)} + (1-p))$, we obtain

$$p^{\alpha_c(p)}(1-p)^{1-\alpha_c(p)}d = \alpha_c(p)^{\alpha_c(p)}(1-\alpha_c(p))^{1-\alpha_c(p)}. \quad (2.18)$$

It is easy to see from Lemma 2.5.1 that $\alpha_c(\cdot)$ is an increasing function on $(0, \frac{1}{d}]$, and from (2.18) that it is strictly increasing.

To complete the proof of Theorem 1.1.8 fix $\varrho \in (0, 1]$. First note (by taking the derivative) that the function $g(p) = p^\varrho(1-p)^{1-\varrho}$ is strictly increasing on the interval $(0, \varrho]$ so that there is indeed a unique solution to the equation characterising p_c . In the special case $\varrho = 1$ this solution is given by $p_c = \frac{1}{d}$. Back to the general case, by (2.18) we have $\varrho = \alpha_c(p_c)$. This value p_c is indeed the critical parameter, since if $p \geq p_c$ we have $\alpha_c(p) \geq \alpha_c(p_c) = \varrho$ so that ϱ -percolation occurs by Lemma 2.5.1. Moreover, if $p < p_c$, then $\alpha_c(p) < \alpha_c(p_c) = \varrho$ so that ϱ -percolation does not occur, which completes the proof of Theorem 1.1.8.

2.6 Minimal supporting subtree for the lattice model

In this section, we will show that in the weak disorder phase the theory developed for the mean field model carries over to the lattice polymer model. In the first part, Section 2.6.1, we will discuss some basic properties of the free energy and its Legendre-Fenchel transform. In the second part, Section 2.6.2, we will develop the ergodic theory for a tree with attached weights, which only requires minor changes from the theory we developed for the mean field model. Finally, in Section 2.6.3, we provide a proof of the existence of a minimal supporting subtree also for the lattice model.

Recall that in the lattice model, the $2d$ -ary tree T comes from the inherent tree structure of the path space. In particular, a vertex v in generation n of this tree corresponds to a directed path on \mathbb{Z}^{1+d} of the form $(j, \omega_j)_{j=1}^n$, where $|\omega_j - \omega_{j-1}| = 1$. In this case, we can write $V(v) = V(n, \omega_n)$ and see T as a tree to whose vertices we attach weights which are no longer independent, but whose dependency structure derives from the underlying lattice model. One feature that remains, however, is that the weights of vertices in distinct generations are independent. It turns that this feature is enough to carry over most of the theory that we developed for the mean field model.

2.6.1 Properties of free energy and its Legendre-Fenchel transform

We start by collecting some of the basic properties of the free energy φ , following e.g. [Com05]. Then, we will use those to present some of the easy bounds on φ . To simplify the notation, we write for $v \in T_n$,

$$H_n(v) = \sum_{j=1}^n V(v_j),$$

which corresponds to the negative of the energy of v . Lower bounds on φ are more difficult than upper bounds. However a very easy lower bound that we will need later on, follows directly from Jensen's inequality

$$\varphi(\beta) = \lim_{n \rightarrow \infty} \mathbb{E} \left[\log \sum_{v \in T_n} e^{\beta \sum_{j=1}^n V(v_j)} \right] \geq \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{v \in T_n} e^{\beta n \mathbb{E} V} = \beta \mathbb{E} V + \log 2d. \quad (2.19)$$

Lemma 2.6.1. *(i) The function $\beta \mapsto \varphi_n(\beta)$ is a smooth function and $\beta \mapsto \varphi(\beta)$ is convex.*

(ii) The function $\beta \mapsto \frac{1}{\beta}(\varphi(\beta) - \log 2d)$ is increasing on $(0, \infty)$.

(iii) The function $\beta \mapsto \frac{1}{\beta} \varphi_n(\beta)$ is decreasing on $(0, \infty)$, and therefore so is $\beta \mapsto \frac{\varphi(\beta)}{\beta}$.

(iv) The function φ is differentiable at 0 with derivative $\varphi'(0) = \mathbb{E} V$.

Proof. Differentiating φ_n yields

$$\frac{d}{d\beta} n\varphi_n(\beta) = \frac{\sum_{v \in T_n} H_n(v) e^{\beta H_n(v)}}{Z_n(\beta)} = E_{\mu_n}[H_n], \quad \frac{d^2}{d\beta} n\varphi_n = \text{Var}_{\mu_n}[H_n] > 0,$$

Hence, φ_n is strictly convex, which implies that φ is convex. In particular, the function $\beta \mapsto \varphi(\beta) - \log(2d)$ is convex and $\varphi(0) = \log 2d$, therefore for $0 < \beta < \beta'$,

$$\varphi(\beta) - \log 2d \leq (1 - \beta/\beta')(\varphi(0) - \log(2d)) + \frac{\beta}{\beta'}(\varphi(\beta') - \log(2d)) = \frac{\beta}{\beta'}(\varphi(\beta') - \log(2d)),$$

which implies the second claim. For (iii) notice that,

$$\frac{d}{d\beta} \frac{\varphi_n(\beta)}{\beta} = -\frac{\varphi_n(\beta)}{\beta^2} + \frac{1}{n\beta} E_{\mu_n}[H_n] = -\frac{1}{n\beta^2} h(\mu_n),$$

where $h(\eta)$ is the entropy of a probability measure

$$h(\eta) := -\sum_{\omega} \eta(\omega) \log \eta(\omega),$$

which is non-negative. Finally for (iv), Jensen's inequality (1.1) and the bound (2.19) imply that

$$0 \leq \varphi(\beta) - \log 2d - \beta \mathbb{E}V \leq \lambda(\beta) - \mathbb{E}V\beta.$$

Since λ is differentiable at 0 with derivative $\lambda'(0) = \mathbb{E}V$, we can divide by $\beta \neq 0$ and so we obtain by letting $\beta \rightarrow 0$ that $\varphi'(0) = \lambda'(0) = \mathbb{E}V$. \square

We can now use the previous lemma to prove an upper bound on φ under certain conditions on the distribution of V . In order to formulate the condition, define the function

$$f(\beta) = \log(2d) + \lambda(\beta) - \beta\lambda'(\beta).$$

Proposition 2.6.2. *If f has a positive root $\beta^f \in (0, \infty)$, then for all $\beta > \beta^f$ we have*

$$\varphi(\beta) \leq \beta \frac{\lambda(\beta^f) + \log(2d)}{\beta^f},$$

so in particular $\varphi(\beta) < \lambda(\beta) + \log(2d)$.

Proof. By Lemma 2.6.1(iii), we find that

$$\mathbb{E}\left[\frac{1}{\beta}\varphi_n(\beta)\right] \leq \inf_{\beta' \in [0, \beta]} \frac{1}{\beta'} \mathbb{E}[\varphi_n(\beta')] \leq \inf_{\beta' \in [0, \beta]} \frac{1}{\beta'} (\lambda(\beta') + \log 2d),$$

where the last inequality follows from Jensen's inequality. Note that the derivative of $\beta' \mapsto \frac{1}{\beta'}(\lambda(\beta') + \log(2d))$ is $-\frac{1}{(\beta')^2} f(\beta')$. So if there exists $\beta^f > 0$ such that $f(\beta^f) = 0$, then the monotonicity of f implies that the optimal $\beta' = \beta^f$, so that the required upper bound follows. \square

We will also discuss some of the properties of the function φ^* that will be useful later on.

Lemma 2.6.3 (Properties of φ^*). *Define $\tilde{\lambda}(\beta) = \lambda(\beta) + \log(2d)$.*

- (i). *For all $\alpha \in \mathbb{R}$, we have $\varphi^*(\alpha) \geq \tilde{\lambda}^*(\alpha)$.*
- (ii). *For all $0 \leq \beta < \beta_c^\varphi$, $\varphi^*(\lambda'(\beta)) = \tilde{\lambda}^*(\lambda'(\beta))$.*
- (iii). *The function $\alpha \mapsto \varphi^*(\alpha)$ is convex and achieves its minimum at $\mathbb{E}V$, where $\varphi^*(\mathbb{E}V) = -\log 2d$. Consequently, φ^* is increasing on $[\mathbb{E}V, \alpha^+]$.*
- (iv). *The function φ^* is strictly convex on a neighbourhood of $\mathbb{E}V$, and so it is even strictly increasing on $[\mathbb{E}V, \alpha^+]$.*
- (v). *For $\alpha \in [\alpha^-, \alpha^+]$, we find that $\varphi^*(\alpha) \leq 0$.*
- (vi). *We have that $\lim_{\alpha \rightarrow \alpha^+} (\varphi^*)'(\alpha) = \infty$.*

Proof. (i) Recall that by Jensen's inequality (1.1), $\varphi(\beta) \leq \tilde{\lambda}(\beta)$ for all $\beta \in \mathbb{R}$. Then, clearly

$$\varphi^*(\alpha) = \sup_{\beta \in \mathbb{R}} (\alpha\beta - \varphi(\beta)) \geq \sup_{\beta \in \mathbb{R}} (\alpha\beta - \tilde{\lambda}(\beta)) = \tilde{\lambda}^*(\alpha).$$

(ii) First note that by the definition of the Legendre-Fenchel transform

$$\varphi^*(\lambda'(\beta)) = \sup_{s \in \mathbb{R}} (\lambda'(\beta)s - \varphi(s)) = s'\lambda'(\beta) - \varphi(s'),$$

where s' is chosen such that $\varphi'_-(s) \leq \lambda'(\beta) \leq \varphi'_+(s)$. Here φ'_- and φ'_+ denote the left and right derivatives of φ respectively. But by definition $\varphi(\beta) = \tilde{\lambda}(\beta)$ for all $0 \leq \beta < \beta_c^\varphi$. In particular, $\varphi(\beta)$ is differentiable at β , and since $\varphi'(\beta) = \lambda'(\beta) = (\tilde{\lambda})'(\beta)$ we have $\varphi^*(\lambda'(\beta)) = \tilde{\lambda}^*(\lambda'(\beta))$.

(iii) By the properties of Legendre-Fenchel transforms, φ^* defines a convex function. It remains to show that it achieves its minimum at $\mathbb{E}V$. Recall from (2.19) that by Jensen's inequality $\varphi(\beta) \geq \beta\mathbb{E}V + \log 2d$, hence

$$\varphi^*(\mathbb{E}V) = \sup_{s \in \mathbb{R}} (s\mathbb{E}V - \varphi(s)) \leq -\log 2d.$$

Also, $\varphi^*(\mathbb{E}V) \geq -\varphi(0) = -\log 2d$, so we get that $\varphi^*(\mathbb{E}V) = -\log 2d$. But, for any $\alpha \in \mathbb{R}$, we have that $\varphi^*(\alpha) = \sup_{\beta \in \mathbb{R}} (\alpha\beta - \varphi(\beta)) \geq -\varphi(0) = -\log 2d$. Hence φ^* achieves its minimum at $\mathbb{E}V$.

(iv) By Lemma 2.6.1, φ is differentiable at 0, and by Proposition 1.1.13 φ is also strictly convex. Thus the Legendre-Fenchel duality implies that φ^* is strictly convex on a neighbourhood of $\mathbb{E}V$.

(v) From the proof of Lemma 1.1.10, we know that for $\beta \geq 0$, $\varphi(\beta) \geq \alpha^+ \beta$ and similarly for $\beta < 0$ we can show that $\varphi(\beta) \geq \alpha^- \beta$. Hence, for $\alpha \in [\alpha^-, \alpha^+]$,

$$\varphi^*(\alpha) = \sup_{\beta \in \mathbb{R}} (\alpha\beta - \varphi(\beta)) \leq \max \left\{ \sup_{\beta < 0} ((\alpha - \alpha^-)\beta), \sup_{\beta \geq 0} ((\alpha - \alpha^+)\beta) \right\} = 0.$$

(vi) Suppose for contradiction that $\beta' := \lim_{\alpha \rightarrow \alpha^+} (\varphi^*)'(\alpha) < \infty$. Then, for $\beta > \beta'$, the function $g : (\alpha^-, \alpha^+) \rightarrow \mathbb{R} : \alpha \mapsto \beta\alpha - \varphi^*(\alpha)$ has strictly positive derivative. Hence by the Legendre-Fenchel duality,

$$\varphi(\beta) = \sup_{\alpha \in [\alpha^-, \alpha^+]} (\alpha\beta - \varphi^*(\alpha)) = \alpha^+ \beta - \varphi^*(\alpha^+).$$

Therefore, φ is linear on (β', ∞) , which contradicts the strict convexity of φ , see Proposition 1.1.13. \square

2.6.2 Ergodic Theory on Weighted Trees

Our main advantage in the weak disorder regime is that the martingale, corresponding to the normalized partition function, has a positive limit. As for the mean-field model this allows us to construct the infinite-volume Gibbs measure $\mu^{(\beta)}$ on the path space. This measure turns out to be the \mathbb{P} -almost sure limit of the polymer measure as $n \rightarrow \infty$.

Let ∂T denote the boundary of the $2d$ -ary tree, i.e. the set of all non-intersecting paths in T which is in one-to-one correspondence with all the (simple-random walk) paths on \mathbb{Z}^d . We give this set a metric space structure, by defining for two paths $\xi, \eta \in \partial T$ the distance

$$d(\xi, \eta) = \exp\{-\inf\{n \geq 0 : \xi_{n+1} \neq \eta_{n+1}\}\},$$

which turns ∂T into a compact metric space.

For a vertex $v = (j, \xi_j)_{j=1}^n \in T_n$, let $B(v) = \{\omega \in \partial T : \omega_j = \xi_j \forall j \leq n\}$. In other words, $B(v)$ consists of all the paths that agree with the simple random walk path (ξ_j) up to the n th step. These sets form a basis open set for the topology induced by the above metric, in particular if we want to define a measure on ∂T , it suffices to specify it on a set of the form $B(v)$. Also, let $T(v)$ be the tree corresponding all the paths in $B(v)$. Note, if w is in the first generation of $T(v)$ it corresponds to a path of length $|v| + 1 = n + 1$ which agrees with v up to the n th step. Then we can define the infinite-volume Gibbs measure

$$\mu^{(\beta)}(B(v)) := e^{\beta \sum_{j=1}^n V(v_j) - n(\lambda(\beta) + \log(2d))} \frac{M^{(\beta)}(v)}{M^{(\beta)}} =: \mu^{(\beta)}(B(v))$$

where $M^{(\beta)}(v)$ is defined as the \mathbb{P} -a.s. limit of

$$M_n^{(\beta)}(v) = \sum_{w \in T(v), |w|=n} \exp\left(\beta \sum_{j=1}^n V(w_j) - n(\lambda(\beta) + \log(2d))\right),$$

which exists since $M_n^{(\beta)}(v)$ defines a positive martingale with respect to the filtration $(\mathcal{G}(|v| + j), j \geq 0)$. Now, we can consider the finite-volume Gibbs measure $\mu_n^{(\beta)}$ as a measure on paths in ∂T , by choosing the first steps of the paths according to $\mu_n^{(\beta)}$ as defined in the introduction, and then extending it uniformly to an infinite paths. With this extension, we see that \mathbb{P} -a.s. for $|v| = k$

$$\mu_n^{(\beta)}(B(v)) = \frac{1}{Z_n(\beta)} e^{\beta \sum_{j=1}^k V(v_j)} \sum_{w \in T_{n-k}} e^{\beta \sum_{j=k+1}^n V(v_j)} \rightarrow \mu^{(\beta)}(B(v)),$$

in other words \mathbb{P} -a.s. $\mu_n^{(\beta)}$ converges weakly to $\mu^{(\beta)}$.

The central result of this section is the following proposition.

Proposition 2.6.4. *Given $\beta < \beta_c$, we have for \mathbb{P} -almost every environment and $\mu^{(\beta)}$ -almost every path ω ,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n V(j, \omega_j) = \lambda'(\beta)$$

and the local dimension of $\mu^{(\beta)}$ is

$$\lim_{n \rightarrow \infty} \frac{-1}{n} \log \mu^{(\beta)}(B(\omega_n)) = \log(2d) + \lambda(\beta) - \beta \lambda'(\beta).$$

Let $\mathcal{V} = (V(v) : v \in T)$ be the environment of weights added to the $2d$ -ary tree T as described at the beginning of the section. We denote for a vertex $x \in T$ by $\mathcal{V}(x) = (V(v) : v \in T(x))$ the space of weights added to the tree $T(x)$. Let **SpinedTrees** be the space of weights attached to the vertices of the $2d$ -ary tree with marked spine, i.e.

$$\mathbf{SpinedTrees} = \{(\mathcal{V}, \xi) \mid \xi \in \partial T\},$$

endowed with the product topology. There is a canonical shift $\theta : \mathbf{SpinedTrees} \rightarrow \mathbf{SpinedTrees}$ which maps (\mathcal{V}, ξ) to the shifted environment $\mathcal{V}(\xi_1)$ together with the trace (ξ_1, ξ_2, \dots) of the spine in the underlying tree. Our aim is to show that θ is a measure-preserving transformation with respect to the measure

$$\nu(d\mathcal{V} d\xi) = \mu_{\mathcal{V}}^{(\beta)}(d\xi) M_{\mathcal{V}}^{(\beta)} \mathbb{P}(d\mathcal{V}),$$

where the subscript \mathcal{V} indicates the dependence of $\mu^{(\beta)}$ and $M^{(\beta)}$ on the underlying environment.

Proof of Proposition 2.6.4. Notice that in the proof of Lemma 2.2.2, we only use the independence of weights in different generations of the tree, therefore the proof carries through in the lattice case and we know that the shift θ is ν -preserving. Similarly, the proof that the shift is ergodic works with a minor change as follows.

Again, by Corollary 16.6 in [LP09], the shift is ergodic with respect to the measure ν

if and only if every set A of weights satisfying

$$\sum_{\substack{\mathcal{V}(v) \in A \\ |v|=1}} \mu_{\mathcal{V}}^{(\beta)}(B(v)) = \mathbb{I}_A(\mathcal{V}) \quad \mathbb{P}\text{-almost surely} \quad (2.20)$$

has $\mathbb{P}(A) \in \{0, 1\}$. Therefore, let A be a set satisfying (2.20), then in particular,

$$\mathcal{V} \in A \iff \mathcal{V}(v) \in A \text{ for all } v \text{ such that } |v| = 1. \quad (2.21)$$

Write the weights $V(j, x)$, $\|x\|_1 \leq j, j \in \mathbb{N}$ as V_1, V_2, \dots such that the corresponding first (directed) index is increasing. Then, given an event $I \in \mathcal{I}$, by iteration the condition (2.21) implies that I is a tail event corresponding to the tail σ -algebra $\bigcap_{j \geq 1} \sigma(V_j)$. Invoking Kolmogorov's zero-one law, we can deduce that $\mathbb{P}(I) = 0$ or 1 , as required.

Since θ is ν -preserving and ergodic, the pointwise ergodic theorem gives us that for \mathbb{P} -a.e. \mathcal{V} and $\mu_{\mathcal{V}}^{(\beta)}$ -a.e. $\omega \in \partial T$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n V(\omega_j) = \nu[V(\omega_1)],$$

where $\nu[\cdot]$ denotes the expectation with respect to the measure ν . Evaluating this expectation yields

$$\begin{aligned} \nu[V(\omega_1)] &= \int V(\omega_1) \mu_{\mathcal{V}}^{(\beta)}(d\omega) M_{\mathcal{V}} \mathbb{P}(d\mathcal{V}) = \int \sum_{|v|=1} V(v) \mu_{\mathcal{V}}^{(\beta)}\{\omega_1 = v\} M_{\mathcal{V}} \mathbb{P}(d\mathcal{V}) \\ &= \int \sum_{|v|=1} V(v) e^{\beta V(v) - \lambda(\beta) - \log(2d)} M_{\mathcal{V}}(v) \mathbb{P}(d\mathcal{V}) \\ &= \int \sum_{|v|=1} V(v) e^{\beta V(v) - \lambda(\beta) - \log(2d)} \mathbb{E}[M_{\mathcal{V}}(v) | V(v)] \mathbb{P}(d\mathcal{V}) \\ &= \frac{\mathbb{E}[V e^{\beta V}]}{\mathbb{E}[e^{\beta V}]} = \lambda'(\beta), \end{aligned}$$

where we used that $\mathbb{E}[M_{\mathcal{V}}(v) | V(v)] = \mathbb{E}[M_{\mathcal{V}}] = 1$. Hence, we have proved the first part of Theorem 2.6.4.

Similarly as before, for the second part of Theorem 2.6.4, notice that

$$\lim_{n \rightarrow \infty} \frac{-1}{n} \log \mu_{\mathcal{V}}^{(\beta)}(B(\xi_n)) = \log(2d) + \lambda(\beta) - \lim_{n \rightarrow \infty} \beta \frac{1}{n} \sum_{j=1}^n V(\xi_j) - \lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{M_{\mathcal{V}}^{(\beta)}(\xi_n)}{M_{\mathcal{V}}^{(\beta)}}. \quad (2.22)$$

Hence by the first part of Theorem 2.6.4, it suffices to show that the second limit

converges to 0. Again, Lemma 2.2.4 applies, so take $g(\mathcal{V}, \xi) = M_{\mathcal{V}}^{(\beta)}(\xi_0)$, then

$$\begin{aligned} g - \theta g &= \log M_{\mathcal{V}}^{(\beta)}(\xi_0) - \log M_{\mathcal{V}}^{(\beta)}(\xi_1) = \log \frac{1}{\mu_{\mathcal{V}}^{(\beta)}(B(\xi_1))} + \beta V(\xi_1) - \lambda(\beta) - \log(2d) \\ &\geq \beta V(\xi_1) - \lambda(\beta) - \log(2d), \end{aligned}$$

where the latter is integrable. Hence by the ergodic theorem and Lemma 2.2.4, for \mathbb{P} -a.e. environment \mathcal{V} and $\mu_{\mathcal{V}}^{(\beta)}$ -a.e. ξ ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{M_{\mathcal{V}}^{(\beta)}(\xi_n)}{M_{\mathcal{V}}^{(\beta)}} = \lim_{n \rightarrow \infty} \frac{-1}{n} \sum_{j=1}^n \log \frac{M_{\mathcal{V}}^{(\beta)}(\xi_{j-1})}{M_{\mathcal{V}}^{(\beta)}(\xi_j)} = -\nu[\log M_{\mathcal{V}}^{(\beta)} - \log M_{\mathcal{V}}^{(\beta)}(\xi_1)] = 0.$$

Therefore (2.22) together with the first part of Theorem 2.6.4 implies the second part of Theorem 2.6.4. \square

2.6.3 Minimal supporting subtree for lattice polymers

In this section, we will prove Theorem 1.1.16(a). The proof of the existence of a minimal supporting subtree for the free energy follows from the ergodic theory developed in the previous Section 2.6.2 as for the mean field model. The fact that any smaller tree is not enough is slightly more subtle on the lattice.

Proof of Theorem 1.1.16(a). The proof follows along the same lines as for the mean field model. By Proposition 2.6.4, we know that for $\mu^{(\beta)}$ -a.e. $\xi \in \partial T$:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n V(\xi_j) = \lambda'(\beta), \quad (2.23)$$

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \mu^{(\beta)}(B(\xi_n)) = f(\beta) = \log(2d) + \lambda(\beta) - \lambda'(\beta)\beta. \quad (2.24)$$

Now let $\varepsilon > 0$, by Egorov's Theorem applied to the convergence in (2.23) and (2.24) we obtain a closed set $A_\varepsilon \subset \partial T$, with $\mu^{(\beta)}(A_\varepsilon) \geq 1 - \varepsilon$ such that the above convergence is uniform on A_ε . Thus, we can choose a sequence $(\delta_n)_{n \in \mathbb{N}}$ with $\delta_n \downarrow 0$, such that for all $\xi \in A_\varepsilon$ and for all $n \geq 1$,

$$\frac{1}{n} \sum_{j=1}^n V(\xi_j) \geq \lambda'(\beta) - \delta_n \quad \text{and} \quad \mu^{(\beta)}(B(\xi_n)) \geq e^{-n(f(\beta) + \delta_n)}. \quad (2.25)$$

Define, $T^{(\varepsilon)} = \bigcup_{\xi \in A_\varepsilon} \xi$, since A_ε is closed and so compact (being the subspace of a compact set), this defines a tree with $\partial T^{(\varepsilon)} = A_\varepsilon$. By the second part of (2.25), there can be at most $\exp(n(f(\beta) + \delta_n))$ vertices in the n th generation, so

$$\frac{1}{n} \log |T_n^{(\varepsilon)}| \leq f(\beta) + \delta_n.$$

As Lemma 2.3.1 also holds in this setting, we obtain that

$$\frac{1}{n} \log |T_n^{(\varepsilon)}| \rightarrow f(\beta). \quad (2.26)$$

Now that we have shown that the constructed tree has the right size, it remains to show that $T^{(\varepsilon)}$ supports the free energy.. By (2.26) there exists a sequence $\gamma_n \downarrow 0$, such that for all $n \geq 1$,

$$\frac{1}{n} \log |T_n^{(\varepsilon)}| \geq f(\beta) - \gamma_n \quad .$$

Now, it follows from the first part of (2.25) that

$$\frac{1}{n} \log \left(\sum_{v \in T_n^{(\varepsilon)}} e^{\beta \sum_{j=1}^n V(v_j)} \right) \geq \frac{1}{n} \log (e^{n(\lambda'(\beta)\beta - \delta_n)} |T_n^{(\varepsilon)}|) \geq \lambda'(\beta)\beta - \delta_n + f(\beta) - \gamma_n ,$$

which converges to $\beta\lambda'(\beta) + f(\beta) = \log(2d) + \lambda(\beta) = \varphi(\beta)$. Setting $\tilde{T} = T^{(\varepsilon)}$, completes the proof of part (a) of Theorem 1.1.16. \square

We can now proceed with the second part of the proof.

Proof of Theorem 1.1.16(b). Fix $\beta > 0$. Note that if there is an interval on which φ^* is linear with slope β , then the definition of $\alpha(\beta)$ guarantees that $\alpha(\beta)$ is the right end point of that interval. Therefore, since φ^* is differentiable by Proposition 1.1.13, there exists a $\delta > 0$ such that φ^* is strictly convex on $[\alpha(\beta), \alpha(\beta) + \delta]$. Therefore by assumption, we can choose $\rho \in (\alpha(\beta), \alpha(\beta) + \delta)$ with $\rho < \alpha^+$ such that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log |A_n| < -\varphi^*(\rho) < -\varphi^*(\alpha(\beta)) ,$$

since φ^* is strictly increasing on $[\mathbb{E}V, \alpha^+]$ by Lemma 2.6.3. Now, choose $\varepsilon > 0$ small enough such that $\limsup_{n \rightarrow \infty} \frac{1}{n} \log |A_n| \leq -\varphi^*(\rho) - 2\varepsilon$. Hence, for n sufficiently large, $|A_n| \leq e^{n(-\varphi^*(\rho) - 2\varepsilon) + \varepsilon} = e^{n(-\varphi^*(\rho) - \varepsilon)}$. Then, by Proposition 1.1.11 we have that for large n ,

$$\#\{v \in T_n : H_n(v) \geq n\rho\} \geq e^{n(-\varphi^*(\rho) - \varepsilon)} \geq |A_n| , \quad (2.27)$$

Next, order the vertices $v_n^1, \dots, v_n^{(2d)^n}$ in the n th generation of T such that $H_n(v_n^1) \geq H_n(v_n^2) \geq \dots \geq H_n(v_n^{(2d)^n})$. Then, clearly

$$\sum_{v \in A_n} e^{\beta H_n(v)} \leq \sum_{j=1}^{|A_n|} e^{\beta H_n(v_n^j)} \leq \sum_{v \in T_n} \mathbb{I}\{v \in T_n : H_n(v) \geq n\rho\} e^{\beta H_n(v)} ,$$

where the last inequality follows from (2.27). Note that by Lemma 1.1.10, $\max_{v \in T_n} H_n(v) \leq \alpha^+ + \varepsilon$ for large n . Hence, we can write

$$\sum_{v \in T_n} \mathbb{I}\{v \in T_n : H_n(v) \geq n\rho\} e^{\beta H_n(v)} \leq \sum_{j=1}^N \#\{v \in T_n : \alpha_{i-1} \leq \frac{1}{n} H_n(v) \leq \alpha_i + \varepsilon\} e^{\beta n(\alpha_i + \varepsilon)} ,$$

where $\alpha_i = (1 - \frac{i}{N})\rho + \frac{i}{N}\alpha^+$, for $i = 1, \dots, N$ and some fixed N . Now, by Proposition 1.1.11 for n sufficiently large,

$$\#\{v \in T_n : H_n(v) \geq n\alpha_{i-1}\} \leq e^{n(-\varphi^*(\alpha_{i-1}) + \varepsilon)}$$

Combining the previous displays and taking $N > \frac{1}{\varepsilon}$ such that $\alpha_i \leq \alpha_{i-1} + \varepsilon$, we obtain

$$\begin{aligned} \sum_{v \in A_n} e^{\beta H_n(v)} &\leq \sum_{i=1}^N e^{n(\beta\alpha_{i-1} - \varphi^*(\alpha_{i-1}) + (1+2\beta)\varepsilon)} \\ &\leq N \exp \left\{ n \left(\max_{\alpha \in [\rho, \alpha^+]} (\beta\alpha - \varphi^*(\alpha)) + (1+2\beta)\varepsilon \right) \right\}. \end{aligned} \quad (2.28)$$

Since $\rho \geq \mathbb{E}V$ it follows that

$$\max_{\alpha \in [\rho, \alpha^+]} (\beta\alpha - \varphi^*(\alpha)) \leq \max_{\alpha \in [\mathbb{E}V, \alpha^+]} (\beta\alpha - \varphi^*(\alpha)) = \varphi(\beta), \quad (2.29)$$

where the last equality follows from the Legendre-Fenchel duality. We now claim that the inequality in the previous display is in fact a strict inequality. Indeed, suppose equality holds for the first two terms in (2.29) and that $\max_{\alpha \in [\rho, \alpha^+]} (\beta\alpha - \varphi^*(\alpha))$ is maximized at α' . Then,

$$\beta\alpha' - \varphi^*(\alpha') = \varphi(\beta).$$

Now by the properties of the Legendre-Fenchel transform $L(x) = \beta x - \varphi(\beta)$ defines a line of support of φ^* at $\alpha(\beta)$. But the previous display tells us that $\varphi^*(\alpha')$ is on that line and since every convex function lies above all its lines of support, it follows that φ^* is linear on $[\alpha(\beta), \alpha']$ and so in particular on $[\alpha(\beta), \rho]$. However, ρ was chosen such that φ^* is strictly convex on $[\alpha(\beta), \rho]$, which yields a contradiction. Therefore, the inequality in (2.29) is strict, i.e. $\max_{\alpha \in [\rho, \alpha^+]} (\beta\alpha - \varphi^*(\alpha)) < \varphi(\beta)$ and thus can additionally assume that ε is small enough such that

$$\max_{\alpha \in [\rho, \alpha^+]} (\beta\alpha - \varphi^*(\alpha)) + 2(1 + \beta)\varepsilon < \varphi(\beta).$$

Then, for n large enough such that $\frac{1}{n} \log N < \varepsilon$, we can combine the previous display with (2.28) to obtain the required inequality,

$$\frac{1}{n} \log \sum_{v \in A_n} e^{\beta H_n(v)} \leq \frac{1}{n} \log N + \max_{\alpha \in [\rho, \alpha^+]} (\beta\alpha - \varphi^*(\alpha)) + (1 + 2\beta)\varepsilon < \varphi(\beta),$$

which completes the proof of Theorem 1.1.16. \square

Chapter 3

Ageing in the parabolic Anderson model

This chapter is based on joint work with P. Mörters and N. Sidorova.¹

In this chapter, we will present the proofs of our results on the parabolic Anderson model presented in Section 1.2. The three sections in this chapter correspond to our main theorems. In Section 3.1, we concentrate on Theorem 1.2.1 which describes ageing in the weak sense, whereas in Section 3.2 we concentrate on the almost sure asymptotics of the residual lifetime function corresponding to the maximizer of the solution. Finally, in Section 3.3, we prove a functional limit result, Theorem 1.2.6.

3.1 Ageing: a weak limit theorem

This section is devoted to the proof of Theorem 1.2.1. In Section 3.1.1 we show ageing for the two point function of the process $(Z_t : t \geq 0)$ of maximizers of the variational problem Φ_t , using the point process approach which was developed in [HMS08] and extended in [KLMS09]. In Section 3.1.2 we use this and the localisation of the profile in Z_t to complete the proof.

3.1.1 Ageing for the maximizer of the variational problem

In this section, we prove ageing for the two point function of the process $(Z_t : t \geq 0)$, which from now on is chosen to be left-continuous.

Proposition 3.1.1. *Let $\theta > 0$, then $\lim_{t \rightarrow \infty} \text{Prob}\{Z_t = Z_{t+\theta t}\} = I(\theta)$, where $I(\theta) \in (0, 1)$ is given by the formula in Proposition 3.1.4 below.*

¹see also [MOS09].

We abbreviate

$$q = \frac{d}{\alpha - d}.$$

For any $t > 0$ consider the point process Π_t on $\mathbb{R}^d \times \mathbb{R}$ defined in (1.10). Define a locally compact Borel set

$$\widehat{H} = \dot{\mathbb{R}}^{d+1} \setminus (\{(x, y) \in \mathbb{R}^d \times \mathbb{R} : y < -q(1 - \varepsilon)|x|\} \cup \{0\}),$$

where $0 < \varepsilon < \frac{1}{1+\theta}$ and $\dot{\mathbb{R}}^{d+1}$ is the one-point compactification of \mathbb{R}^{d+1} . As in Lemma 6.1. of [KLMS09] one can show that the point process Π_t restricted to the domain \widehat{H} converges in law to a Poisson process Π on \widehat{H} with intensity measure

$$\nu(dx dy) = \frac{\alpha dx dy}{(y + q|x|)^{\alpha+1}}. \quad (3.1)$$

Here, Π_t and Π are random elements of the set of point measures on \widehat{H} , which is given the topology of vague convergence. For more background on point processes and similar arguments, see [HMS08].

Our strategy is to express the condition $Z_t = Z_{t+\theta t}$ in terms of the point process Π_t . In order to be able to bound error functions that appear in our calculations, we have to restrict our attention to the point process Π on a large box. To this end, define the two boxes

$$\begin{aligned} B_N &= \{(x, y) \in \mathbb{R}^d \times [0, \infty) : |x| \leq N, \frac{1}{N} \leq y \leq N\}, \\ \widehat{B}_N &= \{(x, y) \in \widehat{H} : |x| \leq N, y \leq N\}. \end{aligned}$$

Now note that the condition $Z_t = Z_{t+\theta t}$ means that

$$\Phi_{t+\theta t}(z) \leq \Phi_{t+\theta t}(Z_t), \quad (3.2)$$

for all $z \in \mathbb{Z}^d$. We now show that it suffices to guarantee that this condition holds for all z in a sufficiently large bounded box.

Lemma 3.1.2. *Define the event*

$$A(N, t) := \left\{ \left(\frac{Z_t}{r_t}, \frac{\Phi_t(Z_t)}{a_t} \right) \in B_N, \Phi_{t+\theta t}(z) \leq \Phi_{t+\theta t}(Z_t) \forall z \in \mathbb{Z}^d \text{ s.t. } \left(\frac{|z|}{r_t}, \frac{\Phi_t(z)}{a_t} \right) \in \widehat{B}_N \right\}.$$

Then, provided the limit on the right-hand side exists, we find that

$$\lim_{t \rightarrow \infty} \text{Prob}\{Z_t = Z_{t+\theta t}\} = \lim_{N \rightarrow \infty} \lim_{t \rightarrow \infty} \text{Prob}(A(N, t)).$$

Proof. We have the lower bound,

$$\begin{aligned} \text{Prob}\{Z_t = Z_{t+\theta t}\} &\geq \text{Prob}\left\{Z_t = Z_{t+\theta t}, \left(\frac{Z_t}{r_t}, \frac{\Phi_t(Z_t)}{a_t} \right) \in B_N\right\} \\ &\geq \text{Prob}(A(N, t)) - \text{Prob}\left\{\frac{|Z_{t+\theta t}|}{r_t} > N\right\}. \end{aligned}$$

Recall that, by [KLMS09, Lemma 6.2], we have that

$$\left(\frac{Z_t}{r_t}, \frac{\Phi_t(Z_t)}{a_t}\right) \Rightarrow (Y^{(1)}, Y^{(2)}), \quad (3.3)$$

where $(Y^{(1)}, Y^{(2)})$ is a random variable on $\mathbb{R}^d \times [0, \infty)$ with an explicit density. In particular, we find that since $r_{t+\theta t} = (1 + \theta)^{q+1} r_t (1 + o(1))$

$$\lim_{t \rightarrow \infty} \text{Prob}\left\{\frac{|Z_{t+\theta t}|}{r_t} > N\right\} = \text{Prob}\left\{|Y^{(1)}| > \frac{N}{(1+\theta)^{q+1}}\right\},$$

which converges to zero as $N \rightarrow \infty$.

Now, for an upper bound on $\text{Prob}\{Z_t = Z_{t(1+\theta)}\}$ we find that

$$\text{Prob}\{Z_t = Z_{t(1+\theta)}\} \leq \text{Prob}(A(N, t)) + \text{Prob}\left\{\frac{|Z_t|}{r_t} \geq N\right\} + \left(1 - \text{Prob}\left\{\frac{1}{N} \leq \frac{\Phi_t(Z_t)}{a_t} \leq N\right\}\right).$$

As above, using the convergence (3.3) one can show that the limit of the last two summands is zero when taking first $t \rightarrow \infty$ and then $N \rightarrow \infty$, which completes the proof of the lemma. \square

We would like to translate the condition (3.2) into a condition on the point process Π_t . Therefore, we have to express $\Phi_{t+\theta t}(z)$ in terms of $\Phi_t(z)$.

Lemma 3.1.3. *For any $z \in \mathbb{Z}^d$ such that $(\frac{z}{r_t}, \frac{\Phi_t(z)}{a_t}) \in \widehat{B}_N$ and $t\xi(z) \geq |z|$,*

$$\frac{\Phi_{t+\theta t}(z)}{a_t} = \frac{\Phi_t(z)}{a_t} + \frac{q\theta}{1+\theta} \frac{|z|}{r_t} + \delta_\theta\left(t, \frac{|z|}{r_t}, \frac{\Phi_t(z)}{a_t}\right),$$

where the error δ_θ converges to zero as $t \rightarrow \infty$ uniformly. Moreover, almost surely, eventually for all large enough t , for all $z \in \mathbb{Z}^d$ such that $(\frac{z}{r_t}, \frac{\Phi_t(z)}{a_t}) \in \widehat{B}_N$ and $t\xi(z) < |z|$, we have that $\Phi_{t+\theta t}(z) \leq 0$, and such a $z \in \mathbb{Z}^d$ will automatically satisfy (3.2).

Proof. Consider any z such that $(\frac{z}{r_t}, \frac{\Phi_t(z)}{a_t}) \in \widehat{B}_N$ and $t\xi(z) \geq |z|$. Then, using that $r_t = \frac{t}{\log t} a_t$ we obtain

$$\begin{aligned} \frac{\Phi_{t+\theta t}(z)}{a_t} &= \frac{\xi(z)}{a_t} - \frac{1}{a_t t + \theta t} (|z| \log \xi(z) - \eta(z)) \\ &= \frac{\Phi_t(z)}{a_t} + \frac{\theta}{1+\theta} \frac{|z|}{r_t \log t} \log a_t + \frac{\theta}{1+\theta} \left(\frac{|z|}{r_t \log t} \log \frac{\xi(z)}{a_t} - \frac{\eta(z)}{t a_t} \right) \\ &= \frac{\Phi_t(z)}{a_t} + \frac{\theta q}{1+\theta} \frac{|z|}{r_t} + \delta'_\theta\left(t, \frac{z}{r_t}, \frac{\xi(z)}{a_t}\right), \end{aligned} \quad (3.4)$$

where using that $\log a_t = (q + o(1)) \log t$ and $0 \leq \eta(z) \leq |z| \log d$, we can write

$$\delta'_\theta\left(t, \frac{z}{r_t}, \frac{\xi(z)}{a_t}\right) = \frac{\theta}{1+\theta} \left(\frac{|z|}{r_t \log t} \log \frac{\xi(z)}{a_t} + o(1) \frac{|z|}{r_t} \right). \quad (3.5)$$

First of all, we have to show that this expression is of the form $\delta_\theta(t, z/r_t, \Phi_t(z)/a_t)$ for

some suitable error function. With this in mind, using that $a_t t = r_t \log t$, we obtain for z such that $t\xi(z) \geq |z|$

$$\begin{aligned} \frac{\Phi_t(z)}{a_t} &= \frac{\xi(z)}{a_t} - \frac{|z|}{r_t \log t} \log \xi(z) + \frac{\eta(z)}{a_t t} \\ &= \frac{\xi(z)}{a_t} - (q + o(1)) \frac{|z|}{r_t} - \frac{|z|}{r_t \log t} \log \frac{\xi(z)}{a_t} + \frac{\eta(z)}{a_t t}, \\ &= \chi_\rho\left(\frac{\xi(z)}{a_t}\right) - (q + o(1)) \frac{|z|}{r_t}, \end{aligned}$$

where $\chi_\rho(x) = x - \rho \log x$ and $\rho = \frac{|z|}{r_t \log t}$. Note that χ_ρ is strictly increasing on $[\rho, \infty)$ and also that $\xi(z)/a_t > \rho$ is equivalent to $t\xi(z) > |z|$ which is satisfied by assumption. Therefore, we can write

$$\frac{\xi(z)}{a_t} = \chi_\rho^{-1}\left(\frac{\Phi_t(z)}{a_t} + (q + o(1)) \frac{|z|}{r_t}\right),$$

and obtain that the error in (3.5) is of the required form

$$\begin{aligned} \delta'_\theta\left(t, \frac{z}{r_t}, \frac{\xi(z)}{a_t}\right) &= \frac{\theta}{1 + \theta} \left(\frac{|z|}{r_t \log t} \log \chi_\rho^{-1}\left(\frac{\Phi_t(z)}{a_t} + (q + o(1)) \frac{|z|}{r_t}\right) + o(1) \frac{|z|}{r_t} \right) \\ &=: \delta_\theta\left(t, \frac{z}{r_t}, \frac{\Phi_t(z)}{a_t}\right). \end{aligned} \quad (3.6)$$

We now show that this error tends to zero uniformly for all z satisfying $t\xi(z) > |z|$ and $(\frac{z}{r_t}, \frac{\Phi_t(z)}{a_t}) \in \widehat{B}_N$. For a lower bound we first use that $x \log x \geq -e^{-1}$ to obtain

$$\begin{aligned} &\frac{|z|}{r_t \log t} \log \chi_\rho^{-1}\left(\frac{\Phi_t(z)}{a_t} + (q + o(1)) \frac{|z|}{r_t}\right) \\ &\geq \frac{|z|}{r_t \log t} \log \frac{|z|}{r_t \log t} \geq -\frac{1}{\log t} e^{-1} - \frac{\log \log t}{\log t} \frac{|z|}{r_t} \geq -\frac{1}{\log t} e^{-1} - \frac{\log \log t}{\log t} N. \end{aligned}$$

To bound the expression in (3.6) from above note that $\rho = \frac{|z|}{r_t \log t} \leq \frac{N}{\log t}$ and we can thus assume that $\rho < 1$, which implies that for $x > 1$ we find $\chi_1(x) \leq \chi_\rho(x)$. Hence, either

$$\chi_\rho^{-1}\left(\frac{\Phi_t(z)}{a_t} + (q + o(1)) \frac{|z|}{r_t}\right) \leq 1,$$

or we can estimate

$$\chi_\rho^{-1}\left(\frac{\Phi_t(z)}{a_t} + (q + o(1)) \frac{|z|}{r_t}\right) \leq \chi_1^{-1}\left(\frac{\Phi_t(z)}{a_t} + (q + o(1)) \frac{|z|}{r_t}\right) \leq \chi_1^{-1}(N(1 + 2q)).$$

which completes the proof of the first part of the lemma.

For the second part, recall that for all $t > 0$ we have $\Phi_t(Z_t) > 0$, since $\Phi_t(0) > 0$. Suppose $t\xi(z) < |z|$, then $\Phi_t(z) = 0$ and hence $z \neq Z_t$. We want to show that $\Phi_{t+\theta t}(z) \leq 0$ which ensures that z satisfies (3.2). Indeed, if $(t + \theta t)\xi(z) < |z|$, then this

is true as $\Phi_{t+\theta t}(z) = 0$, and otherwise we can estimate as above that

$$\frac{\Phi_{t+\theta t}(z)}{a_t} = \frac{\xi(z)}{a_t} - \frac{q\theta}{1+\theta} \frac{|z|}{r_t} + \tilde{\delta}_\theta\left(t, \frac{|z|}{a_t}, \frac{\Phi_t(z)}{a_t}\right),$$

where $\tilde{\delta}_\theta(t, x, y)$ converges to zero uniformly in $(x, y) \in \widehat{B}_N$. In particular, it follows that

$$\frac{\Phi_{t+\theta t}(z)}{a_t} \leq \left(-\frac{q\theta}{1+\theta} + \frac{1}{\log t}\right) \frac{|z|}{r_t} + \tilde{\delta}_\theta\left(t, \frac{|z|}{a_t}, \frac{\Phi_t(z)}{a_t}\right),$$

which is negative for all t large enough, uniformly for all z such that $(\frac{z}{r_t}, \frac{\Phi_t(z)}{a_t}) \in \widehat{B}_N$. \square

We now calculate $\text{Prob}(A(t, N))$ in the limit as $t \rightarrow \infty$, i.e. we are interested in

$$\iint_{(x,y) \in B_N} \text{Prob}\left\{\frac{Z_t}{r_t} \in dx, \frac{\Phi_t(Z_t)}{a_t} \in dy, \Phi_{t+\theta t}(z) \leq \Phi_{t+\theta t}(Z_t) \forall z \in \mathbb{Z}^d \text{ s.t. } \left(\frac{|z|}{r_t}, \frac{\Phi_t(z)}{a_t}\right) \in \widehat{B}_N\right\}.$$

Before we continue, we need to clarify what we mean by the above notation. We write

$$\int_A \text{Prob}\{X \in dx, Y \in B\},$$

instead of

$$\int \text{Prob}\{Y \in B | X = x\} \text{Prob}_X(dx),$$

where Prob_X is the distribution of X and $\text{Prob}\{Y \in B | X = x\}$ is a regular conditional probability as defined in [Bre68]. The latter exists, see [Bre68, Theorem 4.43], since in our case we take Y to be a point process, i.e. Y takes values in the Polish space of non-negative Radon measures on \mathbb{R}^{d+1} equipped with the vague topology.

First, we express the probability under the integral for fixed $(x, y) \in B_N$ in terms of the point process Π_t . Given that Π_t contains the point (x, y) we require that there are no points in the set $\mathbb{R}^d \times (y, \infty)$, and requiring (3.2) for all points z with $(|z|/r_t, \Phi_t(z)/a_t) \in \widehat{B}_N$ is, by Lemma 3.1.3, equivalent to the requirement that Π_t should have no points in the set

$$\{(\bar{x}, \bar{y}) \in \widehat{B}_N : \bar{y} + \frac{q\theta}{1+\theta} |\bar{x}| > y + \frac{q\theta}{1+\theta} |x|\}.$$

Hence, defining the set

$$D_\theta^N(r, y) = \{(\bar{x}, \bar{y}) \in \mathbb{R}^d \times \mathbb{R} : \bar{y} > y\} \cup \{(\bar{x}, \bar{y}) \in \widehat{B}_N : |\bar{x}| > r, \bar{y} > y - \frac{q\theta}{1+\theta} (|\bar{x}| - r)\},$$

we see that, as $t \rightarrow \infty$,

$$\begin{aligned} \lim_{t \rightarrow \infty} \text{Prob}(A(N, t)) &= \iint_{(x, y) \in B_N} \text{Prob}\{\Pi(\mathrm{d}x, \mathrm{d}y) = 1, \Pi(D_\theta^N(|x|, y)) = 0\} \\ &= \iint_{(x, y) \in B_N} e^{-\nu(D_\theta^N(|x|, y))} \nu(\mathrm{d}x \mathrm{d}y). \end{aligned}$$

Taking the limit in this way is justified as $D_\theta^N(|x|, y)$ is relatively compact in \widehat{H} and (x, y) ranges only over elements in B_N . Finally, if we similarly define

$$D_\theta(r, y) = \{(\bar{x}, \bar{y}) \in \mathbb{R}^d \times \mathbb{R} : |\bar{x}| \leq r, \bar{y} > y \text{ or } |\bar{x}| > r, \bar{y} > y - \frac{q\theta}{1+\theta}(|\bar{x}| - r)\}.$$

we can invoke Lemma 3.1.2 to see that

$$\begin{aligned} \lim_{t \rightarrow \infty} \text{Prob}\{Z_t = Z_{t+\theta t}\} &= \lim_{N \rightarrow \infty} \lim_{t \rightarrow \infty} \text{Prob}(A(N, t)) \\ &= \lim_{N \rightarrow \infty} \iint_{(x, y) \in B_N} e^{-\nu(D_\theta^N(|x|, y))} \nu(\mathrm{d}x \mathrm{d}y) \\ &= \int_{y \geq 0} \int_{x \in \mathbb{R}^d} e^{-\nu(D_\theta(|x|, y))} \nu(\mathrm{d}x \mathrm{d}y), \end{aligned}$$

where the last equality follows by dominated convergence, as the integrand is dominated by $e^{-\nu(D_0(|x|, y))}$ which is integrable with respect to ν by the direct calculation in the next proposition. For an illustration of the region $D_\theta(|x|, y)$ see Figure 3-1.

We now simplify the expression that arises from the point process calculation. We denote by $B(a, b)$ the Beta function with parameters a, b and define the normalized incomplete Beta function

$$\tilde{B}(x, a, b) = \frac{1}{B(a, b)} \int_0^x v^{a-1} (1-v)^{b-1} \mathrm{d}v.$$

Proposition 3.1.4 (Explicit form of $I(\theta)$). *For any $\theta \geq 0$, we have*

$$\int_{y \geq 0} \int_{x \in \mathbb{R}^d} e^{-\nu(D_\theta(|x|, y))} \nu(\mathrm{d}x \mathrm{d}y) = I(\theta) := \frac{1}{B(\alpha - d + 1, d)} \int_0^1 v^{\alpha-d} (1-v)^{d-1} \varphi_\theta(v) \mathrm{d}v,$$

where the weight $\varphi_\theta(v)$ is defined by

$$\frac{1}{\varphi_\theta(v)} = 1 - \tilde{B}(v, \alpha - d, d) + (1 + \theta)^\alpha \left(\frac{\theta}{v} + 1\right)^{d-\alpha} \tilde{B}\left(\frac{v+\theta}{1+\theta}, \alpha - d, d\right). \quad (3.7)$$

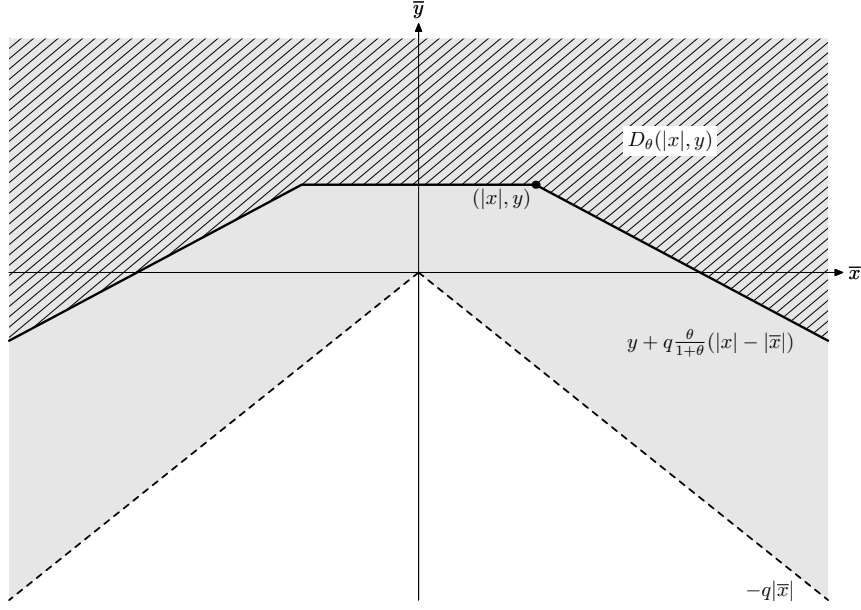


Figure 3-1: The point process Π is defined on the set \hat{H} indicated in grey. If we fix $Z_t/r_t \in dx, \Phi_t(Z_t)/a_t \in dy$, the condition that $Z_t = Z_{t+\theta t}$ corresponds to the requirement that the point process Π has no points in the shaded region $D_\theta(|x|, y)$.

Proof. First of all, we compute $\nu(D_\theta(r, y))$ for some $r > 0$,

$$\begin{aligned} \nu(D_c(r, y)) &= \int_{|\bar{x}| \leq r} \int_y^\infty \frac{\alpha d\bar{x} d\bar{y}}{(\bar{y} + q|\bar{x}|)^{\alpha+1}} + \int_{|\bar{x}| > r} \int_{y - \frac{q\theta}{1+\theta}(|\bar{x}| - r)}^\infty \frac{\alpha d\bar{x} d\bar{y}}{(\bar{y} + q|\bar{x}|)^{\alpha+1}} \\ &= \int_{|\bar{x}| \leq r} \frac{d\bar{x}}{(y + q|\bar{x}|)^\alpha} + \int_{|\bar{x}| > r} \frac{d\bar{x}}{(y + \frac{q\theta}{1+\theta}r + \frac{q}{1+\theta}|\bar{x}|)^\alpha}. \end{aligned}$$

Next, we can rewrite the two last summands. We exploit the invariance of the integrand under reflections at the axes, then for $x_i \geq 0$ we use the substitution $u_1 = \bar{x}_1 + \dots + \bar{x}_d$, $u_i = \bar{x}_i$ for $i \geq 2$ and then the substitution $y + qu_1 = y/v$, so that

$$\begin{aligned} \int_{|\bar{x}| \leq r} \frac{d\bar{x}}{(y + q|\bar{x}|)^\alpha} &= 2^d \int_0^r \frac{u_1^{d-1}}{(y + qu_1)^\alpha} \left(\int_{\substack{u_2 + \dots + u_d \leq 1 \\ u_i \geq 0}} du_2 \dots du_d \right) du_1 \\ &= \frac{2^d}{(d-1)!} \int_0^r \frac{u_1^{d-1}}{(y + qu_1)^\alpha} du_1 = \frac{2^d y^{d-\alpha}}{q^d (d-1)!} \int_{\frac{y}{y+qr}}^1 v^{\alpha-d-1} (1-v)^{d-1} dv \\ &= \vartheta y^{d-\alpha} \left(1 - \tilde{B}\left(\frac{y}{y+qr}, \alpha - d, d\right) \right), \end{aligned} \tag{3.8}$$

where $\vartheta = \frac{2^d B(\alpha-d, d)}{q^d (d-1)!}$. A similar calculation shows that

$$\int_{|\bar{x}| > r} \frac{d\bar{x}}{(y + \frac{q\theta}{1+\theta}r + \frac{q}{1+\theta}|\bar{x}|)^\alpha} = \vartheta (1+\theta)^d \left(y + \frac{q\theta}{1+\theta}r \right)^{d-\alpha} \tilde{B}\left(\frac{y + \frac{q\theta}{1+\theta}r}{y+qr}, \alpha - d, d\right).$$

Combining the previous displays, and using the substitution $y + qr = y/v$ yields

$$\begin{aligned}
& \nu(D_\theta(r, y)) \\
&= \vartheta y^{d-\alpha} \left(1 - \tilde{B}\left(\frac{y}{y+qr}, \alpha - d, d\right) + (1 + \theta)^d \left(1 + \frac{q\theta}{1+\theta} \frac{r}{y}\right)^{d-\alpha} \tilde{B}\left(\frac{y + \frac{q\theta}{1+\theta} r}{y+qr}, \alpha - d, d\right) \right) \\
&= \vartheta y^{d-\alpha} \left(1 - \tilde{B}(v, \alpha - d, d) + (1 + \theta)^\alpha \left(1 + \frac{\theta}{v}\right)^{d-\alpha} \tilde{B}\left(\frac{v+\theta}{1+\theta}, \alpha - d, d\right) \right) \\
&= \vartheta y^{d-\alpha} \varphi_\theta(v)^{-1}.
\end{aligned} \tag{3.9}$$

To calculate the integral over $x \in \mathbb{R}^d$ we substitute $r = x_1 + \dots + x_d$ and $u_i = x_i$ for $i \geq 2$,

$$\int_{\mathbb{R}^d} e^{-\nu(D_\theta(|x|, y))} \frac{\alpha}{(y + q|x|)^{\alpha+1}} dx = \frac{2^d}{(d-1)!} \int_0^\infty e^{-\nu(D_\theta(r, y))} \frac{\alpha r^{d-1}}{(y + qr)^{\alpha+1}} dr.$$

Finally, we integrate over $y \geq 0$ and use the above formula for $\nu(D_\theta(r, y))$ together with the substitution $y + qr = y/v$ and $w = \vartheta y^{d-\alpha}$ to obtain

$$\begin{aligned}
\int_{y \geq 0} \int_{x \in \mathbb{R}^d} e^{-\nu(D_\theta(|x|, y))} \nu(dx dy) &= \frac{2^d}{(d-1)!} \int_0^\infty \int_0^\infty e^{-\nu(D_\theta(r, y))} \frac{\alpha r^{d-1} dr}{(y + qr)^{\alpha+1}} dy \\
&= \frac{2^d}{q^d (d-1)!} \int_0^1 \alpha v^{\alpha-d} (1-v)^{d-1} \int_0^\infty \exp\{-\vartheta y^{d-\alpha} \varphi_\theta(v)^{-1}\} y^{d-\alpha-1} dy dv \\
&= \frac{\alpha}{B(\alpha-d, d)(\alpha-d)} \int_0^1 v^{\alpha-d} (1-v)^{d-1} \int_0^\infty e^{-w \varphi_\theta(v)^{-1}} dw dv \\
&= \frac{1}{B(\alpha-d+1, d)} \int_0^1 v^{\alpha-d} (1-v)^{d-1} \varphi_\theta(v) dv,
\end{aligned}$$

where we used the identity $B(x+1, y)(x+y) = B(x, y)x$ for $x, y > 0$ in the last step. \square

Proposition 3.1.5 (Tails of I). (a) $\lim_{\theta \rightarrow \infty} \theta^d I(\theta) = \frac{1}{dB(\alpha-d+1, d)}$.

(b) $\lim_{\theta \downarrow 0} \theta^{-1}(1 - I(\theta)) = C_0$, where the constant C_0 is given by

$$C_0 = \frac{1}{B(\alpha-d+1, d)} \left(\int_0^1 \alpha v^{\alpha-d} (1-v)^{d-1} \tilde{B}(v; \alpha - d, d) dv + B(2(\alpha - d), 2d - 1) \right).$$

Proof. (a) As $\tilde{B}(v, \alpha - d, d) \leq 1$ and $v \mapsto \tilde{B}(v, \alpha - d, d)$ is nondecreasing we get, for $0 \leq v \leq 1$,

$$\begin{aligned}
\frac{1}{\varphi_\theta(v)} &= 1 - \tilde{B}(v, \alpha - d, d) + (1 + \theta)^\alpha \left(\frac{\theta}{v} + 1\right)^{d-\alpha} \tilde{B}\left(\frac{v+\theta}{1+\theta}, \alpha - d, d\right) \\
&\geq (1 + \theta)^d v^{\alpha-d} \left(\frac{1+\theta}{v+\theta}\right)^{\alpha-d} \tilde{B}\left(\frac{\theta}{1+\theta}, \alpha - d, d\right) \geq \frac{1}{2} (1 + \theta)^d v^{\alpha-d},
\end{aligned}$$

where we chose θ large enough such that $\tilde{B}(\frac{\theta}{1+\theta}, \alpha - d, d) \geq \frac{1}{2}$. Hence, we deduce that

for θ large enough,

$$\begin{aligned}\theta^d I(\theta) &= \frac{1}{B(\alpha-d+1, d)} \int_0^1 v^{\alpha-d} (1-v)^{d-1} \theta^d \varphi_\theta(v) dv \\ &\leq \frac{2}{B(\alpha-d+1, d)} \int_0^1 (1-v)^{d-1} dv = \frac{2}{dB(\alpha-d+1, d)}.\end{aligned}$$

Therefore, since $\theta^d \varphi_\theta(v) \rightarrow v^{d-\alpha}$ pointwise for every $v \in (0, 1)$ as $\theta \rightarrow \infty$, we can invoke the dominated convergence theorem to complete the proof of the lemma.

(b) We can write

$$1 - I(\theta) = \frac{1}{B(\alpha-d+1, d)} \int_0^1 v^{\alpha-d} (1-v)^{d-1} \varphi_\theta(v) (\varphi_\theta^{-1}(v) - 1) dv.$$

Set $\tilde{B}(v) = \tilde{B}(v; \alpha-d, d)$. Then, since $\varphi_\theta(v) \rightarrow 1$ for every v as $\theta \downarrow 0$, we can concentrate on

$$\begin{aligned}\varphi_\theta(v)^{-1} - 1 &= (1+\theta)^d v^{\alpha-d} \left(\frac{1+\theta}{v+\theta}\right)^{\alpha-d} \tilde{B}\left(\frac{\theta+v}{1+\theta}\right) - \tilde{B}(v) \\ &= \tilde{B}\left(\frac{\theta+v}{1+\theta}\right) \left(\frac{(1+\theta)^\alpha v^{\alpha-d}}{(v+\theta)^{\alpha-d}} - 1\right) + \tilde{B}\left(\frac{\theta+v}{1+\theta}\right) - \tilde{B}(v).\end{aligned}$$

The first summand can be bounded by $(1+\theta)^\alpha - 1 \leq 2\alpha\theta$, eventually for all θ . For the second term, we have that

$$\tilde{B}\left(\frac{\theta+v}{1+\theta}\right) - \tilde{B}(v) = \int_v^{\frac{v+\theta}{1+\theta}} u^{\alpha-d-1} (1-u)^{d-1} du \leq \theta(1-v)^d \max\{v^{\alpha-d-1}, 1\}.$$

Combining the two estimates we obtain that $\theta^{-1}(1 - I(\theta))$ is bounded, so that by the dominated convergence theorem, we may take the limit of $\theta^{-1}(\varphi_\theta^{-1}(v) - 1)$ as $\theta \downarrow 0$ under the integral. \square

3.1.2 Ageing for the solution u

In this section, we prove Theorem 1.2.1 by combining the results about ageing for the maximizer Z_t from the previous section with the localisation results in [KLMS09]. We start with a preliminary calculation that will be used several times in the remainder.

Lemma 3.1.6. *If $\Phi_t(x) = \Phi_t(y)$ for some $t > 0$ and $x, y \in \mathbb{Z}^d$ such that $t\xi(x) > |x|$ and $t\xi(y) > |y|$, then for all $s > 0$ such that $s\xi(x) > |x|$ and $s\xi(y) > |y|$, we have that*

$$\Phi_s(x) - \Phi_s(y) = (\xi(x) - \xi(y)) \left(1 - \frac{t}{s}\right).$$

Proof. By the assumptions on t, x, y , we find that

$$\Phi_t(x) - \Phi_t(y) = (\xi(x) - \xi(y)) - \frac{1}{t} (|x| \log \xi(x) - |y| \log \xi(y) - \eta(x) + \eta(y)) = 0.$$

Rearranging, we can substitute into

$$\begin{aligned}\Phi_s(x) - \Phi_s(y) &= (\xi(x) - \xi(y)) - \frac{1}{s}(|x| \log \xi(x) - |y| \log \xi(y) - \eta(x) + \eta(y)) \\ &= (\xi(x) - \xi(y)) \left(1 - \frac{t}{s}\right),\end{aligned}$$

which completes the proof. \square

Remark 3.1.7. Let $Z_t^{(1)}, Z_t^{(2)}, \dots \in \mathbb{Z}^d$ be sites in \mathbb{Z}^d producing the largest values of Φ_t in descending order (choosing the site with largest ℓ^1 -norm in case of a tie), and recall that $Z_t = Z_t^{(1)}$. It is then easy to see that $t\xi(Z_t^{(i)}) > |Z_t^{(i)}|$ for $i = 1, 2$ and all $t \geq 1$. Hence, if $\tau > 1$ is a jump time of the process $(Z_t: t > 0)$, then $\Phi_\tau(Z_\tau^{(1)}) = \Phi_\tau(Z_\tau^{(2)})$, so that we can apply Lemma 3.1.6 with $x = Z_\tau^{(1)}$ and $y = Z_\tau^{(2)}$ and the conclusion holds for all $s \geq \tau$.

Lemma 3.1.8. *Almost surely, the function $u \mapsto \xi(Z_u)$ is nondecreasing on $(1, \infty)$.*

Proof. Let $\{\tau_n\}$ be the successive jump times of the process $(Z_t: t \geq 1)$. By definition,

$$\Phi_{\tau_{n+1}}(Z_{\tau_{n+1}}^{(1)}) = \Phi_{\tau_{n+1}}(Z_{\tau_{n+1}}^{(2)})$$

and by right-continuity of $t \mapsto Z_t^{(1)}$, we have that $Z_{\tau_{n+1}}^{(2)} = Z_{\tau_n}^{(1)}$. Now, consider $\tau_{n+1} < t < \tau_{n+2}$ such that $Z_t^{(i)} = Z_{\tau_{n+1}}^{(i)}$ for $i = 1, 2$, then by Lemma 3.1.6 and Remark 3.1.7 we know that

$$\begin{aligned}\Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(2)}) &= \Phi_t(Z_{\tau_{n+1}}^{(1)}) - \Phi_t(Z_{\tau_{n+1}}^{(2)}) = (\xi(Z_{\tau_{n+1}}^{(1)}) - \xi(Z_{\tau_{n+1}}^{(2)}))(1 - \frac{\tau_{n+1}}{t}) \\ &= (\xi(Z_{\tau_{n+1}}^{(1)}) - \xi(Z_{\tau_n}^{(1)}))(1 - \frac{\tau_{n+1}}{t}).\end{aligned}\tag{3.10}$$

As $t < \tau_{n+2}$, and $t \mapsto \Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(2)})$ is not constant, the left hand side of (3.10) is strictly positive, which implies that $\xi(Z_{\tau_{n+1}}) - \xi(Z_{\tau_n}) > 0$, thus completing the proof. \square

As an immediate consequence of this lemma, we get that $(Z_t: t > 1)$ never returns to the same point in \mathbb{Z}^d . We now prove the first part of Theorem 1.2.1.

Lemma 3.1.9. *For any sufficiently small $\varepsilon > 0$,*

$$\lim_{t \rightarrow \infty} \text{Prob}\left\{ \sup_{z \in \mathbb{R}^d} |v(t, z) - v(t + \theta t, z)| < \varepsilon \right\} = \lim_{t \rightarrow \infty} \text{Prob}\{Z_t = Z_{t+\theta t}\} = I(\theta).$$

Proof. Suppose $0 < \varepsilon < \frac{1}{2}$ and let us throughout this proof argue on the event

$$A_t = \left\{ v(t, Z_t) > 1 - \frac{\varepsilon}{2}, v(t + \theta t, Z_{t+\theta t}) > 1 - \frac{\varepsilon}{2} \right\}.$$

Now, if $z \neq Z_t$, then

$$u(t, z) \leq \sum_{x \neq Z_t} u(t, x) = U(t) - u(t, Z_t) < \frac{\varepsilon}{2} U(t),$$

and similarly if $z \neq Z_{t+\theta t}$, then $u(t+\theta t, z) \leq \frac{\varepsilon}{2} U(t+\theta t)$. In particular, if $z \neq Z_t$ and $z \neq Z_{t+\theta t}$, then $|v(t, z) - v(t+\theta t, z)| < \varepsilon$. Now, if $Z_t = Z_{t+\theta t}$, then by assumption A_t we have $|v(t, z) - v(t+\theta t, z)| < \varepsilon$ for any $z \in \mathbb{Z}^d$. Conversely, suppose that $Z_t \neq Z_{t+\theta t}$. From above we then get $u(t+\theta t, Z_t) < \frac{\varepsilon}{2} U(t+\theta t)$ and since we argue on the event A_t , we find that $v(t, Z_t) - v(t+\theta t, Z_t) > 1 - \varepsilon > \varepsilon$, so that

$$\sup_{z \in \mathbb{Z}^d} |v(t, z) - v(t+\theta t, z)| \geq |v(t, Z_t) - v(t+\theta t, Z_t)| > \varepsilon.$$

To complete the proof, it remains to notice that since $v(t, Z_t)$ converges weakly to one, we have that $\text{Prob}(A_t) \rightarrow 1$ as $t \rightarrow \infty$. \square

Before we can prove the remaining part of Theorem 1.2.1, we need to collect the following fact about the maximizers $Z^{(1)}$ and $Z^{(2)}$.

Lemma 3.1.10. *Let $\lambda_t = (\log t)^{-\beta}$ for some $\beta > 1 + \frac{1}{\alpha-d}$. If $t_1 \leq t_2$ are sufficiently large, satisfy $Z_{t_1}^{(1)} = Z_{t_2}^{(1)}$ and*

$$\Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(2)}) \geq \frac{1}{2} a_t \lambda_t, \quad (3.11)$$

holds for $t = t_1$ and $t = t_2$, then (3.11) holds for all $t \in [t_1, t_2]$.

Proof. First, we additionally assume that $Z_t^{(2)} = Z_{t_1}^{(2)}$ for all $t \in [t_1, t_2]$. By Lemma 3.1.8 we have that $Z_t^{(1)} = Z_{t_1}^{(1)}$ for all $t \in [t_1, t_2]$. Using also the continuity of $t \mapsto \Phi_t(Z_t^{(i)})$, $i = 1, 2$, we get

$$\begin{aligned} \Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(2)}) &= \Phi_t(Z_{t_1}^{(1)}) - \Phi_t(Z_{t_1}^{(2)}) \\ &= \xi(Z_{t_1}^{(1)}) - \xi(Z_{t_1}^{(2)}) - \frac{1}{t} (|Z_{t_1}^{(1)}| \log \xi(Z_{t_1}^{(1)}) - |Z_{t_1}^{(2)}| \log \xi(Z_{t_1}^{(2)}) - \eta(Z_{t_1}^{(1)}) + \eta(Z_{t_1}^{(2)})) \\ &= A - \frac{1}{t} B \quad \text{for all } t \in [t_1, t_2], \end{aligned}$$

for some constants $A, B \in \mathbb{R}$ depending only on t_1 . Now, defining

$$f(t) = A - \frac{1}{t} B - \frac{1}{2} a_t \lambda_t,$$

we get that $f(t_1) \geq 0$ and $f(t_2) \geq 0$ by our assumption. Moreover,

$$f'(t) = \frac{1}{t^2} \left(B - \frac{1}{2} \frac{t^{q+1}}{(\log t)^{q+\beta}} \left(q - \frac{q+\beta}{\log t} \right) \right),$$

which is negative for t larger than some threshold depending on t_1 . Also, if t_1 is large enough, the function $t \mapsto \frac{t^{q+1}}{(\log t)^{q+\beta}} \left(q - \frac{q+\beta}{\log t} \right)$ is strictly increasing for $t \geq t_1$, hence f' has at most one zero for $t \geq t_1$. Therefore, if f' has a zero $t' \geq t_1$, then f' is negative for all $t > t'$, implying that f does not have a minimum at $t' \in (t_1, t_2)$. If f' does not have a zero for $t \geq t_1$, then it follows that $f'(t) < 0$ for all $t \geq t_1$. In either case, $f(t_1) \geq 0$ and $f(t_2) \geq 0$ imply that $f(t) \geq 0$ for all $t \in [t_1, t_2]$, in other words (3.11) holds for all $t \in [t_1, t_2]$.

Now we drop the extra assumption on $Z_t^{(2)}$ and define the jump times

$$\tau^- = \sup \{t < t_1 : Z_t^{(1)} \neq Z_{t_1}^{(1)}\} \quad \text{and} \quad \tau^+ = \inf \{t > t_2 : Z_t^{(1)} \neq Z_{t_1}^{(1)}\}.$$

Furthermore, define a sequence $s^{(i)}$ by setting $s^{(0)} = \tau^-$ and for $i \geq 1$ setting

$$s^{(i)} = \inf \{s > s^{(i-1)} : \Phi_s(Z_s^{(2)}) = \Phi_s(Z_s^{(3)})\}.$$

Then, there exists $N \geq 1$ such that $s^{(N)} < \tau^+ < s^{(N+1)}$, where $N \geq 1$ since, by Lemma 3.1.8,

$$Z_{\tau^-}^{(2)} = \lim_{t \uparrow \tau^-} Z_t^{(1)} \neq Z_{\tau^+}^{(1)}.$$

Using that $\Phi_{s^{(i)}}(Z_{s^{(i)}}^{(2)}) = \Phi_{s^{(i)}}(Z_{s^{(i)}}^{(3)})$ and Proposition 3.4 in [KLMS09],

$$\Phi_{s^{(i)}}(Z_{s^{(i)}}^{(1)}) - \Phi_{s^{(i)}}(Z_{s^{(i)}}^{(2)}) = \Phi_{s^{(i)}}(Z_{s^{(i)}}^{(1)}) - \Phi_{s^{(i)}}(Z_{s^{(i)}}^{(3)}) \geq a_{s^{(i)}} \lambda_{s^{(i)}}.$$

Therefore, (3.11) holds for $t = s^{(i)}$, $i = 1, \dots, N$. Hence, the additional assumption that we made in the first part of the proof holds for each of the intervals $[t_1, s^{(1)})$, $[s^{(i)}, s^{(i+1)})$, for $i = 1, \dots, N-1$ and $[s^{(N)}, t_2)$. Thus, we can deduce that (3.11) holds for all t in the union of these intervals, which completes the proof. \square

Finally, we can now show the stronger form of ageing for the profile v and thereby complete the proof of Theorem 1.2.1.

Proof of Theorem 1.2.1. By Proposition 3.1.1, it suffices to show that

$$\lim_{t \rightarrow \infty} \text{Prob} \left\{ \sup_{\substack{z \in \mathbb{R}^d \\ s \in [t, t+\theta t]}} |v(t, z) - v(s, z)| < \varepsilon \right\} = \lim_{t \rightarrow \infty} \text{Prob} \{Z_t^{(1)} = Z_{t+\theta t}^{(1)}\}.$$

First of all, note that by Lemma 3.1.8 we know that $Z_t^{(1)} = Z_{t+\theta t}^{(1)}$ if and only if $Z_t^{(1)} = Z_s^{(1)}$ for all $s \in [t, t+\theta t]$. We will work on the event

$$A_t = \{\Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(2)}) \geq a_t \lambda_t / 2\} \cap \{\Phi_{t+\theta t}(Z_{t+\theta t}^{(1)}) - \Phi_{t+\theta t}(Z_{t+\theta t}^{(2)}) \geq a_{t+\theta t} \lambda_{t+\theta t} / 2\}.$$

Recall from Proposition 5.3 in [KLMS09] that if $\Phi_t(Z_t^{(1)})$ and $\Phi_t(Z_t^{(2)})$ are sufficiently far apart, then the profile is localized in $Z_t^{(1)}$. More precisely, almost surely,

$$\lim_{t \rightarrow \infty} \sum_{z \in \mathbb{Z}^d \setminus \{Z_t^{(1)}\}} v(t, z) \mathbb{I}\{\Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(2)}) \geq a_t \lambda_t / 2\} = 0.$$

In particular, for given $\varepsilon < \frac{1}{2}$, we can assume that t is sufficiently large, so that for all $s \geq t$,

$$\sum_{z \in \mathbb{Z}^d \setminus \{Z_s^{(1)}\}} v(s, z) \mathbb{I}\{\Phi_s(Z_s^{(1)}) - \Phi_s(Z_s^{(2)}) \geq a_s \lambda_s / 2\} < \frac{\varepsilon}{2}. \quad (3.12)$$

Now, if $Z_t^{(1)} \neq Z_{t+\theta t}^{(1)}$, then on A_t , we know by (3.12) that $v(t+\theta t, Z_t^{(1)}) \leq \frac{\varepsilon}{2}$. Combining

this with the fact that $v(t, Z_t^{(1)}) > 1 - \frac{\varepsilon}{2}$, we have that

$$\sup_{\substack{z \in \mathbb{Z}^d \\ s \in [t, t+\theta t]}} |v(t, z) - v(s, z)| \geq |v(t, Z_t^{(1)}) - v(t + \theta t, Z_t^{(1)})| > 1 - \varepsilon > \varepsilon.$$

Conversely, assume that $Z_t^{(1)} = Z_{t+\theta t}^{(1)}$, then by Lemma 3.1.8, $Z_t^{(1)} = Z_s^{(1)}$ for all $s \in [t, t + \theta t]$. Now, on the event A_t we know by Lemma 3.1.10 that for all $s \in [t, t + \theta t]$,

$$\Phi_s(Z_s^{(1)}) - \Phi_s(Z_s^{(2)}) \geq a_s \lambda_s / 2. \quad (3.13)$$

This implies by (3.12) that

$$\sum_{z \in \mathbb{Z}^d \setminus \{Z_s^{(1)}\}} v(s, z) < \varepsilon / 2 \quad \text{for all } s \in [t, t + \theta t].$$

As in the proof of Lemma 3.1.9, this yields that

$$\sup_{\substack{z \in \mathbb{R}^d \\ s \in [t, t+\theta t]}} |v(t, z) - v(s, z)| < \varepsilon.$$

Hence, to complete the proof, it remains to notice that by [KLMS09, Lemma 6.2] the pair $(\Phi_t(Z_t^{(1)})/a_t, \Phi_t(Z_t^{(2)})/a_t)$ converges weakly to a limit random variable with a density, from which we conclude that $\text{Prob}(A_t) \rightarrow 1$ as $t \rightarrow \infty$. \square

3.2 Ageing: an almost-sure limit theorem

In this section, we prove Theorem 1.2.3. As in the previous section, we first concentrate on an analogous theorem for the maximizer of the variational problem Φ_t . In particular, in Section 3.2.1, we extend Proposition 3.1.1 to a moderate deviations principle. This estimate allows us to prove the equivalent of the almost sure ageing Theorem 1.2.3 in the setting of the variational problem in Section 3.2.2. Finally, in Section 3.2.3, we transfer this result to the maximizer of v .

3.2.1 Moderate deviations

Recall from Proposition 3.1.5 that

$$\lim_{t \rightarrow \infty} \text{Prob}\{Z_t = Z_{t+\theta t}\} = I(\theta) \sim \frac{1}{dB(\alpha - d + 1, d)} \theta^{-d},$$

where the latter asymptotic equivalence holds for θ tending to infinity. We now show that we obtain the same asymptotics for $\text{Prob}\{Z_t = Z_{t+\theta t}\}$ if we allow θ to grow slowly with t .

Proposition 3.2.1 (Moderate deviations). *For any positive function θ_t such that $\theta_t \rightarrow \infty$ and $\theta_t \leq (\log t)^\delta$ for some $\delta > 0$, we have that*

$$\text{Prob}\{Z_t = Z_{t(1+\theta_t)}\} = \left(\frac{1}{dB(\alpha-d+1, d)} + o(1)\right) \theta_t^{-d}.$$

Unlike in the proof of Proposition 3.1.1, we cannot directly use the point process techniques, as the weak convergence only applies to compact sets, whereas here we deal with sets that increase slowly with t to a set that has infinite mass under the intensity measure ν . We start by expressing $\Phi_t(z)$ in terms of $\xi(z)$ and $|z|$, while carefully controlling the errors.

Lemma 3.2.2. *For z such that $t\xi(z) > |z|$, we find that*

$$\frac{\xi(z)}{a_t} - q \frac{|z|}{r_t} \left(1 + 2 \frac{\log(N_t + qg_t)}{\log t}\right) \leq \frac{\Phi_t(z)}{a_t} \leq \frac{\xi(z)}{a_t} - q \frac{|z|}{r_t} \left(1 - C_1 \frac{\log \log t}{\log t}\right) + C_2 \frac{1}{\log t},$$

for some constants $C_1, C_2 > 0$, where the upper bound holds uniformly for all z , whereas the lower bound holds uniformly for all z such that $\Phi_t(z) \leq a_t N_t$ and $|z| \leq r_t g_t$ for any functions N_t, g_t such that $N_t, g_t \rightarrow \infty$ as $t \rightarrow \infty$. Similarly, for $\theta \geq 0$ and for all z such that $(1 + \theta)t\xi(z) > |z|$,

$$\frac{\xi(z)}{a_t} - \frac{q}{1 + \theta} \frac{|z|}{r_t} \left(1 + 2 \frac{\log(N_t + qg_t)}{\log t}\right) \leq \frac{\Phi_{t+\theta t}(z)}{a_t} \leq \frac{\xi(z)}{a_t} - \frac{q}{1 + \theta} \frac{|z|}{r_t} \left(1 - C_1 \frac{\log \log t}{\log t}\right) + C_2 \frac{1}{\log t},$$

for some constants $C_1, C_2 > 0$ (not depending on θ), again with the restriction that for the lower bound we assume that $\Phi_t(z) \leq a_t N_t$ and $|z| \leq r_t g_t$.

Proof. Using that $r_t = \frac{t}{\log t} a_t$, we have, for $t\xi(z) > |z|$, that

$$\begin{aligned} \frac{\Phi_t(z)}{a_t} &= \frac{\xi(z)}{a_t} - \frac{1}{ta_t} (|z| \log \xi(z) - \eta(z)) \\ &= \frac{\xi(z)}{a_t} - q \frac{|z|}{r_t} + q \frac{|z|}{r_t} \frac{\log \log t}{\log t} - \frac{|z|}{r_t \log t} \log \frac{\xi(z)}{a_t} + \frac{\eta(z)}{r_t \log t}. \end{aligned} \quad (3.14)$$

It thus suffices to find suitable upper and lower bounds for the last two terms. For the upper bound, we use that $\eta(z) \leq |z| \log d$ and also that $x \log x \geq -e^{-1}$ for any $x > 0$, to get

$$-\frac{|z|}{r_t \log t} \log \frac{\xi(z)}{a_t} + \frac{\eta(z)}{a_t t} \leq -\frac{|z|}{r_t \log t} \log \frac{|z|}{r_t \log t} + \frac{|z| \log d}{r_t \log t} \leq \frac{1}{\log t} e^{-1} + \frac{|z| \log \log t + \log d}{r_t \log t},$$

so that the upper bound holds if $C_1 \geq 3 + q$ and $C_2 = e^{-1}$. Similarly, for the lower bound we first note that by the above calculation

$$\frac{\Phi_t(z)}{a_t} + q \frac{|z|}{r_t} \geq \frac{\xi(z)}{a_t} - \frac{|z|}{r_t \log t} \log \frac{\xi(z)}{a_t}.$$

Now, either $\xi(z)/a_t < (1 + \frac{g_t}{\log t})^2$ or if $\xi(z)/a_t \geq (1 + \frac{g_t}{\log t})^2$ we can estimate using that

$\log x \leq x^{1/2}$ for all $x > 0$

$$\frac{\xi(z)}{a_t} - \frac{|z|}{r_t \log t} \log \frac{\xi(z)}{a_t} \geq \frac{\xi(z)}{a_t} - \frac{|z|}{r_t \log t} \left(\frac{\xi(z)}{a_t} \right)^{1/2} \geq \frac{\xi(z)}{a_t} - \frac{g_t}{\log t} \left(\frac{\xi(z)}{a_t} \right)^{1/2} \geq \left(\frac{\xi(z)}{a_t} \right)^{1/2}.$$

Therefore, we have

$$\frac{\xi(z)}{a_t} \leq \max \left\{ \left(1 + \frac{g_t}{\log t} \right)^2, \left(\frac{\Phi_t(z)}{a_t} + q \frac{|z|}{r_t} \right)^2 \right\} \leq (N_t + qg_t)^2.$$

Hence, we can conclude by (3.14) that

$$\frac{\Phi_t(z)}{a_t} \geq \frac{\xi(z)}{a_t} - q \frac{|z|}{r_t} \left(1 + \frac{2 \log(N_t + qg_t)}{\log t} \right).$$

For the bound on $\Phi_{t+\theta t}$, we write

$$\begin{aligned} \frac{\Phi_{t+\theta t}(z)}{a_t} &= \frac{\xi(z)}{a_t} - \frac{1}{(1+\theta)ta_t} (|z| \log \xi(z) - \eta(z)) \\ &= \frac{\xi(z)}{a_t} - \frac{q}{1+\theta} \frac{|z|}{r_t} + \frac{1}{1+\theta} \left(q \frac{|z|}{r_t} \frac{\log \log t}{\log t} - \frac{|z|}{r_t \log t} \log \frac{\xi(z)}{a_t} + \frac{\eta(z)}{r_t \log t} \right) \\ &= \frac{\xi(z)}{a_t} - \frac{q}{1+\theta} \frac{|z|}{r_t} + \frac{1}{1+\theta} \text{error}(t, z), \end{aligned}$$

where $\text{error}(t, z)$ is exactly the same error term that we controlled in the first part of the lemma and $\frac{1}{1+\theta} \leq 1$, so that the proof of the lemma is complete. \square

In analogy to the proof of Proposition 3.1.1, we will have to restrict $(Z_t/r_t, \Phi_t(Z_t)/a_t)$ to large boxes in $\mathbb{R}^d \times \mathbb{R}$. The first step is therefore to estimate the probability that $(Z_t/r_t, \Phi_t(Z_t)/a_t)$ lies outside a large box.

Lemma 3.2.3. *There exist constants $C, C' > 0$ such that for all $t > 0$ large enough, uniformly for all $N \geq 1$,*

$$(a) \text{ Prob} \left\{ \frac{|Z_t|}{r_t} \geq N \right\} \leq C N^{d-\alpha}.$$

$$(b) \text{ Prob} \left\{ \frac{\Phi_t(Z_t)}{a_t} \geq N \right\} \leq C N^{d-\alpha}.$$

(c) *For any positive function η_t bounded by 1 such that $\eta_t a_t \rightarrow \infty$*

$$\text{Prob} \left\{ \frac{\Phi_t(Z_t)}{a_t} \leq \eta_t \right\} \leq C e^{-C' \eta_t^{d-\alpha}}.$$

Proof. (a) Using Lemma 3.2.2, we can estimate

$$\begin{aligned} \text{Prob} \{ |Z_t| \geq N r_t \} &\leq \text{Prob} \{ \text{there exists } |z| \geq N r_t : \frac{\Phi_t(z)}{a_t} \geq 0 \} \\ &\leq \sum_{|z| \geq N r_t} \text{Prob} \left\{ \frac{\xi(z)}{a_t} \geq q \frac{|z|}{r_t} \left(1 - C_1 \frac{\log \log t}{\log t} \right) - C_2 \frac{1}{\log t} \right\} \\ &= (1 + o(1)) \sum_{|z| \geq N r_t} a_t^{-\alpha} \left(q \frac{|z|}{r_t} \right)^{-\alpha} = (1 + o(1)) q^{-\alpha} r_t^{\alpha-d} \sum_{|z| \geq N r_t} |z|^{-\alpha}, \end{aligned}$$

where we used that $r_t^d = a_t^\alpha$ and we write $o(1)$ for a function that tends to 0 as $t \rightarrow \infty$ (uniformly in $N \geq 1$). Approximating the sum by an integral we get the upper bound

$$\begin{aligned} & (1 + o(1))q^{-\alpha}r_t^{\alpha-N} \int_{|z| \geq Nr_t} (|z| - 1)^{-\alpha} dz \\ &= (1 + o(1)) \frac{2^d}{(d-1)!q^\alpha} r_t^{\alpha-d} \int_{r \geq r_t N} r^{d-1} (r - 1)^{-\alpha} dr \leq (1 + o(1)) \frac{2^{d+\alpha}}{(d-1)!q^\alpha} N^{d-\alpha}, \end{aligned}$$

assuming that $r_t \geq 2$, so that the first claim follows.

(b) For the second estimate, we use again Lemma 3.2.2 to obtain

$$\begin{aligned} \text{Prob}\{\Phi_t(Z_t) \geq Na_t\} &\leq \sum_{z \in \mathbb{Z}^d} \text{Prob}\{\Phi_t(z) \geq Na_t\} \\ &\leq \sum_{z \in \mathbb{Z}^d} \text{Prob}\left\{\frac{\xi(z)}{a_t} \geq N + q \frac{|z|}{r_t} \left(1 - C_1 \frac{\log \log t}{\log t}\right) - C_2 \frac{1}{\log t}\right\} \\ &\leq (1 + o(1)) \sum_{z \in \mathbb{Z}^d} a_t^{-\alpha} \left(N + q \frac{|z|}{r_t}\right)^{-\alpha}. \end{aligned}$$

Similarly as before, and assuming t is large enough such that $r_t \geq 2q$, we can estimate the sum by an integral and finally use the substitution $\frac{N}{2} + qr = \frac{N}{2} \frac{1}{v}$ to bound the above display by

$$\begin{aligned} &\leq (1 + o(1))r_t^{\alpha-d} \int_{z \in \mathbb{R}^d} (r_t N + q(|z| - 1))^{-\alpha} \\ &\leq (1 + o(1)) \frac{2^d}{(d-1)!} \int_{r > 0} r^{d-1} \left(\frac{1}{2}N + qr\right)^{-\alpha} \\ &= (1 + o(1)) \frac{2^\alpha}{q^d(d-1)!} N^{d-\alpha} \int_0^1 v^{\alpha-d-1} (1-v)^{d-1} dv \\ &= (1 + o(1)) \frac{2^\alpha}{q^d(d-1)!} B(\alpha - d, d) N^{d-\alpha}. \end{aligned}$$

(c) For the last bound, note first that by Lemma 3.2.2, that if $t\xi(z) > |z|$ and $|z|/r_t < g_t := \log t$ and $\Phi_t(z)/a_t < 1$, then there exists $C > 0$ such that

$$\frac{\xi(z)}{a_t} - q \frac{|z|}{r_t} \left(1 + C \frac{\log \log t}{\log t}\right) < \frac{\Phi_t(z)}{a_t}.$$

Hence, we can estimate

$$\begin{aligned} \text{Prob}\left\{\frac{\Phi_t(Z_t)}{a_t} \leq \eta_t\right\} &\leq \text{Prob}\left\{\text{for all } z \text{ s.t. } t\xi(z) > |z| \text{ and } |z| < r_t(\log t) : \frac{\Phi_t(z)}{a_t} \leq \eta_t\right\} \\ &\leq \prod_{|z| \leq r_t g_t} \text{Prob}\left\{t\xi(z) \leq |z| \text{ or } \frac{\xi(z)}{a_t} \leq \eta_t + q \frac{|z|}{r_t} \left(1 + C \frac{\log \log t}{\log t}\right)\right\}. \end{aligned}$$

Now, if $t\xi(z) \leq |z|$, then since $r_t \log t = ta_t$ and since we can assume $\log t \geq 1/q$, we have that

$$\frac{\xi(z)}{a_t} \leq \frac{|z|}{r_t \log t} \leq \eta_t + q \frac{|z|}{r_t} \left(1 + C \frac{\log \log t}{\log t}\right).$$

Combining with the above we obtain

$$\begin{aligned} \text{Prob}\left\{\frac{\Phi_t(Z_t)}{a_t} \leq \eta_t\right\} &\leq \prod_{|z| \leq r_t g_t} \text{Prob}\left\{\frac{\xi(z)}{a_t} \leq \eta_t + q \frac{|z|}{r_t} \left(1 + C \frac{\log \log t}{\log t}\right)\right\} \\ &= \exp\left\{\sum_{|z| \leq r_t g_t} \log\left(1 - a_t^{-\alpha} \left(\eta_t + q \frac{|z|}{r_t} \left(1 + C \frac{\log \log t}{\log t}\right)\right)^{-\alpha}\right)\right\}. \end{aligned}$$

Since $a_t \eta_t \rightarrow \infty$, we can use that $\log(1 - x) \leq -x$ for $x < 1$, and continue by

$$\begin{aligned} &\leq \exp\left\{-(1 + o(1)) \sum_{|z| \leq r_t g_t} a_t^{-\alpha} \left(\eta_t + q \frac{|z|}{r_t}\right)^{-\alpha}\right\} \\ &\leq \exp\left\{-(1 + o(1)) r_t^{-d} \frac{2^d}{(d-1)!} \int_0^{r_t g_t - \frac{1}{2}} r^{d-1} \left(\eta_t + q \frac{r + \frac{1}{2}}{r_t}\right)^{-\alpha} dr\right\} \\ &= \exp\left\{-(1 + o(1)) \frac{2^d}{(d-1)!} \left(1 + \frac{1}{2} q \eta_t^{-1} r_t^{-1}\right)^{-\alpha} \int_0^{g_t - \frac{1}{2} r_t^{-1}} r^{d-1} (\eta_t + qr)^{-\alpha} dr\right\}. \end{aligned}$$

Again using the substitution $\eta_t + qr = \eta_t \frac{1}{v}$, we can write

$$\int_0^{g_t - \frac{1}{2} r_t^{-1}} r^{d-1} (\eta_t + qr)^{-\alpha} dr = q^{-d} \eta_t^{d-\alpha} \int_0^1 \mathbb{I}\left\{v \geq \frac{\eta_t}{\eta_t + q(g_t - \frac{1}{2} r_t^{-1})}\right\} v^{\alpha-d-1} (1-v)^{d-1} dv.$$

Finally, since $\eta_t \leq 1$ and $g_t \rightarrow \infty$, the latter integral converges to $B(\alpha - d, d)$ and the claim follows. \square

Proof of Proposition 3.2.1. As in the proof of Proposition 3.1.1, the *main idea* is to restrict $(Z_t/r_t, \Phi_t(Z_t)/a_t)$ to large boxes to be able to control the error when approximating Φ_t . To set up the notation, we introduce functions $\eta_t = (\log t)^{-\beta'}$; $N_t = (\log t)^\beta$; $g_t = (\log t)^\gamma$ for some parameters $\beta, \beta', \gamma > 0$, which we will choose later on depending on the function θ_t . In particular, the idea is to restrict the maximizer such that

$$\text{Prob}\{Z_t = Z_{t(1+\theta_t)}\} = \text{Prob}\{Z_t = Z_{t(1+\theta_t)}; |Z_t| \leq r_t g_t; \Phi_t(z)/a_t \in [\eta_t, N_t]\} + o(\theta_t^{-d}).$$

Once these growing boxes are defined, we can find by Lemma 3.2.2 a constant $C > 0$ such that the function $\delta_t = C \frac{\log \log t}{\log t}$ satisfies

$$\frac{\xi(z)}{a_t} - q \frac{|z|}{r_t} (1 + \delta_t) \leq \frac{\Phi_t(z)}{a_t} \leq \frac{\xi(z)}{a_t} - q \frac{|z|}{r_t} (1 - \delta_t) + \delta_t,$$

where the upper bound holds for all z and the lower bound for all z such that $|z| \leq r_t g_t$ and $\Phi_t(z) \leq a_t N_t$.

Upper bound. We use a slight variation on the general idea, and consider

$$\begin{aligned} \text{Prob}\{Z_t = Z_{t(1+\theta_t)}\} &\leq \text{Prob}\left\{Z_t = Z_{t(1+\theta_t)}; \eta_t \leq \frac{\xi(Z_t)}{a_t} - q \frac{|Z_t|}{r_t} (1 - \delta_t) + \delta_t < N_t\right\} \\ &\quad + \text{Prob}\{\Phi_t(Z_t) < \eta_t a_t\} + \sum_{z \in \mathbb{Z}^d} \text{Prob}\left\{\frac{\xi(z)}{a_t} - q \frac{|z|}{r_t} (1 - \delta_t) + \delta_t < N_t\right\}. \end{aligned} \quad (3.15)$$

But by Lemma 3.2.3(a) and the proof of (b), we have that

$$\text{Prob}\{\Phi_t(Z_t) < \eta_t a_t\} + \sum_{z \in \mathbb{Z}^d} \text{Prob}\left\{\frac{\xi(z)}{a_t} - q \frac{|z|}{r_t} (1 - \delta_t) + \delta_t < N_t\right\} \leq C_1 (e^{-C_2 \eta_t^{d-\alpha}} + N_t^{d-\alpha}),$$

so that this error term is of order $o(\theta_t^{-d})$ by assuming that N_t grows fast enough.

Now, we can unravel the definition of Z_t being the maximizer of Φ_t (in particular we know $t\xi(Z_t) > |Z_t|$ and $\Phi_t(Z_t)$ is positive) and write

$$\begin{aligned} &\text{Prob}\{Z_t = Z_{t+\theta_t}; \eta'_t a_t \leq \Phi_t(Z_t) \leq \eta_t a_t; |Z_t| \leq g_t r_t\} \\ &= \int_{\eta_t}^{N_t} \sum_{z \in \mathbb{Z}^d} \text{Prob} \left\{ \begin{array}{ll} \Phi_t(\bar{z}) \leq \Phi_t(z) & \text{for } \bar{z}: t\xi(\bar{z}) > |\bar{z}|; \\ \Phi_{t(1+\theta_t)}(\bar{z}) \leq \Phi_{t(1+\theta_t)}(z) & \text{for } \bar{z}: t(1+\theta_t)\xi(\bar{z}) > |\bar{z}|; \\ t\xi(z) > |z|; & \frac{\xi(z)}{a_t} - q \frac{|z|}{r_t} (1 - \delta_t) + \delta_t \in dy \end{array} \right\}. \end{aligned} \quad (3.16)$$

We continue by finding an upper bound for the latter probability. Let z be such that $|z| < g_t r_t$, and $\frac{\xi(z)}{a_t} - q \frac{|z|}{r_t} (1 - \delta_t) + \delta_t = y < N_t$. Then for any \bar{z} such that $|\bar{z}| < g_t r_t$ satisfying

$$\left\{ \begin{array}{ll} \Phi_t(\bar{z}) \leq \Phi_t(z) & \text{if } t\xi(\bar{z}) > |\bar{z}| \\ \Phi_{t(1+\theta_t)}(\bar{z}) \leq \Phi_{t(1+\theta_t)}(z) & \text{if } t(1+\theta_t)\xi(\bar{z}) > |\bar{z}| \end{array} \right\},$$

we can deduce by Lemma 3.2.2, and recalling that $\delta_t = C \log \log t / \log t$,

$$\left\{ \begin{array}{ll} \frac{\xi(\bar{z})}{a_t} - q \frac{|\bar{z}|}{r_t} (1 + \delta_t) \leq y & \text{if } t\xi(\bar{z}) > |\bar{z}| \\ \frac{\xi(\bar{z})}{a_t} - \frac{q}{1+\theta_t} \frac{|\bar{z}|}{r_t} (1 + \delta_t) \leq y + \frac{q\theta_t}{1+\theta_t} \frac{|z|}{r_t} (1 - \delta_t) & \text{if } t(1+\theta_t)\xi(\bar{z}) > |\bar{z}| \end{array} \right\}.$$

Now, if $t\xi(\bar{z}) \leq |\bar{z}|$, then since $r_t \log t = ta_t$ and since we can assume $\log t \geq 1/q$, we also have that

$$\frac{\xi(\bar{z})}{a_t} \leq \frac{|\bar{z}|}{r_t \log t} \leq y + q \frac{|\bar{z}|}{r_t} (1 + \delta_t).$$

Similarly, if $(t + \theta_t t)\xi(\bar{z}) \leq |\bar{z}|$, then

$$\frac{\xi(\bar{z})}{a_t} \leq \frac{|\bar{z}|}{(1 + \theta_t)r_t \log t} \leq y + \frac{q\theta_t}{1 + \theta_t} \frac{|z|}{r_t} (1 - \delta_t) + \frac{q|\bar{z}|}{(1 + \theta_t)r_t} (1 + \delta_t).$$

Therefore using the independence of the $\xi(z)$, we get an upper bound on the expression

in (3.16)

$$\begin{aligned} & \int_{\eta_t}^{N_t} \sum_{z \in \mathbb{Z}^d} \text{Prob}\left\{\frac{\xi(z)}{a_t} - q \frac{|z|}{r_t} (1 - \delta_t) + \delta_t \in dy\right\} \prod_{|\bar{z}| < |z|} \text{Prob}\left\{\frac{\xi(\bar{z})}{a_t} \leq y + q \frac{|\bar{z}|}{r_t} (1 + \delta_t)\right\} \\ & \times \prod_{|z| < |\bar{z}| < r_t g_t} \text{Prob}\left\{\frac{\xi(\bar{z})}{a_t} - \frac{q}{1+\theta_t} \frac{|\bar{z}|}{r_t} (1 + \delta_t) \leq y + \frac{q\theta_t}{1+\theta_t} \frac{|z|}{r_t} (1 - \delta_t)\right\}. \end{aligned} \quad (3.17)$$

So far, we have not imposed any restrictions on the exponent $\beta' > 0$ in the definition of $\eta_t = (\log t)^{\beta'}$, so that we can now require that $\eta_t \delta_t \rightarrow 0$, i.e that $\beta' < 1$. In the following steps, we treat each of the products in the above expression separately. First of all, since $\text{Prob}\{\xi(0) \in dy\} = \alpha y^{-\alpha-1} dy$, we have that

$$\begin{aligned} \text{Prob}\left\{\frac{\xi(z)}{a_t} - q \frac{|z|}{r_t} (1 - \delta_t) + \delta_t \in dy\right\} &= \alpha a_t^{-\alpha} \left(y + q \frac{|z|}{r_t} (1 - \delta_t) - \delta_t\right)^{-(\alpha+1)} dy \\ &\leq (1 - \delta_t \eta_t^{-1})^{-(\alpha+1)} \alpha a_t^{-\alpha} \left(y + q \frac{|z|}{r_t}\right)^{-(\alpha+1)} dy, \end{aligned}$$

where by our assumption on η_t , we have that $\delta_t \eta_t^{-1} = o(1)$. For the second expression in (3.17), we find that for all $y > \eta_t$, we know that $a_t y > a_t \eta_t > 1$, assuming that t is large enough. In particular, we can use the approximation $\log(1 - x) < -x$ for $x < 1$ to obtain uniformly for all $y > \eta_t$ and all z ,

$$\begin{aligned} \prod_{|\bar{z}| < |z|} \text{Prob}\left\{\frac{\xi(\bar{z})}{a_t} \leq y + q \frac{|\bar{z}|}{r_t} (1 + \delta_t)\right\} &\leq \exp\left\{\sum_{|\bar{z}| < |z|} \log\left(1 - a_t^{-\alpha} \left(y + q \frac{|\bar{z}|}{r_t} (1 + \delta_t)\right)^{-\alpha}\right)\right\} \\ &\leq \exp\left\{-\sum_{|\bar{z}| < |z|} a_t^{-\alpha} \left(y + q \frac{|\bar{z}|}{r_t} (1 + \delta_t)\right)^{-\alpha}\right\} \\ &\leq \exp\left\{-(1 + \delta_t)^{-\alpha} \int_{|\bar{z}| < |z| - \frac{1}{2}} r_t^{-d} \left(y + q \frac{|\bar{z}| + \frac{1}{2}}{r_t}\right)^{-\alpha} dz\right\} \\ &\leq \exp\left\{-(1 + \delta_t)^{-\alpha} \left((1 + q\eta_t^{-1} r_t^{-1})^{-\alpha} \int_{|\bar{x}| < \frac{|z|}{r_t}} (y + q|\bar{x}|)^{-\alpha} d\bar{x} - r_t^{-d} \eta_t^{-\alpha}\right)\right\} \\ &= (1 + o(1)) \exp\left\{-(1 + o(1)) \int_{|\bar{x}| < \frac{|z|}{r_t}} (y + q|\bar{x}|)^{-\alpha} d\bar{x}\right\}, \end{aligned}$$

where our assumptions on η_t guarantee that all the error terms are of order $o(1)$. Finally, we consider the last product in (3.17), and a similar calculation to above shows that uniformly in $y \geq \eta_t$ and for all $z \in \mathbb{Z}^d$,

$$\begin{aligned} \prod_{|z| < |\bar{z}| < r_t g_t} \text{Prob}\left\{\frac{\xi(\bar{z})}{a_t} - \frac{q}{1+\theta_t} \frac{|\bar{z}|}{r_t} (1 + \delta_t) \leq y + \frac{q\theta_t}{1+\theta_t} \frac{|z|}{r_t} (1 - \delta_t)\right\} \\ \leq \exp\left\{-(1 + \delta_t)^{-\alpha} \sum_{|z| < |\bar{z}| < r_t g_t} r_t^{-d} \left(y + \frac{q\theta_t}{1+\theta_t} \frac{|z|}{r_t} + \frac{q}{1+\theta_t} \frac{|\bar{z}|}{r_t}\right)^{-\alpha}\right\} \\ \leq (1 + o(1)) \exp\left\{-(1 + o(1)) \int_{\frac{|z|}{r_t} \leq |\bar{x}| \leq g_t} \left(y + \frac{q\theta_t}{1+\theta_t} \frac{|z|}{r_t} + \frac{q}{1+\theta_t} |\bar{x}|\right)^{-\alpha} d\bar{x}\right\}. \end{aligned}$$

Combining these estimates to bound (3.17) and thus (3.16), we obtain

$$\begin{aligned}
& \text{Prob}\{Z_t = Z_{t+\theta_t}; \eta'_t a_t \leq \frac{\xi(Z_t)}{a_t} - q \frac{|Z_t|}{r_t} (1 - \delta_t) + \delta_t \leq \eta_t a_t; |Z_t| \leq g_t r_t\} \\
& \leq (\alpha + o(1)) \int_{\eta_t}^{N_t} \sum_{z \in \mathbb{Z}^d} r_t^{-d} \exp \left\{ - (1 + o(1)) \int_{|\bar{x}| < \frac{|z|}{r_t}} (y + q|\bar{x}|)^{-\alpha} d\bar{x} \right\} \\
& \quad \times \exp \left\{ - (1 + o(1)) \int_{\frac{|z|}{r_t} \leq |\bar{x}| \leq g_t} \left(y + \frac{q\theta_t}{1+\theta_t} \frac{|z|}{r_t} + \frac{q}{1+\theta_t} |\bar{x}| \right)^{-\alpha} d\bar{x} \right\} (y + q \frac{|z|}{r_t})^{-(\alpha+1)} dy \\
& \leq (1 + o(1)) \int_{\eta_t}^{N_t} \int_{x \in \mathbb{R}^d} \exp \left\{ - (1 + o(1)) \int_{|\bar{x}| < |x|} (y + q|\bar{x}|)^{-\alpha} d\bar{x} \right\} \\
& \quad \times \exp \left\{ - (1 + o(1)) \int_{|x| \leq |\bar{x}| \leq g_t} \left(y + \frac{q\theta_t}{1+\theta_t} |x| + \frac{q}{1+\theta_t} |\bar{x}| \right)^{-\alpha} d\bar{x} \right\} \frac{\alpha dx dy}{(y + q|x|)^{\alpha+1}},
\end{aligned}$$

where as before the latter approximation works since $\eta_t a_t \rightarrow \infty$. Note also that uniformly in x and y

$$\int_{|\bar{x}| \geq g_t} \left(y + \frac{q\theta_t}{1+\theta_t} |x| + \frac{q}{1+\theta_t} |\bar{x}| \right)^{-\alpha} d\bar{x} \leq (1 + \theta_t)^\alpha q^{-\alpha} \int_{|\bar{x}| \geq g_t} |\bar{x}|^{-\alpha} \leq C' \theta_t^\alpha g_t^{d-\alpha},$$

where $C' > 0$ is some universal constant. We thus have to additionally assume that g_t grows fast enough such that this term tends to 0. Hence, together with (3.15) we have shown that

$$\begin{aligned}
& \text{Prob}\{Z_t = Z_{t(1+\theta_t)}\} \\
& \leq (1 + o(1)) \int_{y>0} \int_{x \in \mathbb{R}^d} \exp \left\{ - (1 + o(1)) \int_{|\bar{x}| < |x|} (y + q|\bar{x}|)^{-\alpha} d\bar{x} \right\} \\
& \quad \exp \left\{ - (1 + o(1)) \int_{|x| \leq |\bar{x}|} \left(y + \frac{q\theta_t}{1+\theta_t} |x| + \frac{q}{1+\theta_t} |\bar{x}| \right)^{-\alpha} d\bar{x} \right\} \frac{\alpha dx dy}{(y + q|x|)^{\alpha+1}} + o(\theta_t^{-d}).
\end{aligned}$$

Lower bound. Before we simplify the expression for the upper bound, we derive a similar expression for the lower bound. As in the upper bound, we follow the main idea and restrict our attention to large boxes and estimate

$$\begin{aligned}
& \text{Prob}\{Z_t = Z_{t(1+\theta_t)}\} \geq \sum_{|z| \leq r_t g_t} \text{Prob}\{Z_t = z = Z_{t+\theta_t}; \frac{\xi(z)}{a_t} - 2q \frac{|z|}{r_t} \leq N_t\} \\
& = \sum_{|z| \leq r_t g_t} \text{Prob} \left\{ \begin{array}{ll} \Phi_t(\bar{z}) \leq \Phi_t(z) & \text{for } \bar{z}: t\xi(\bar{z}) > |\bar{z}|; \\ \Phi_{t(1+\theta_t)}(\bar{z}) \leq \Phi_{t(1+\theta_t)}(z) & \text{for } \bar{z}: t(1+\theta_t)\xi(\bar{z}) > |\bar{z}|; \\ t\xi(z) > |z|; & \frac{\xi(z)}{a_t} - 2q \frac{|z|}{r_t} \leq N_t \end{array} \right\}.
\end{aligned} \tag{3.18}$$

The proof of Lemma 3.2.2 shows that if z is such that $|z| \leq g_t r_t$ and $\frac{\xi(z)}{a_t} - 2q \frac{|z|}{r_t} \leq N_t$,

then we can find $C > 0$ such that with $\delta_t = C \frac{\log \log t}{\log t}$ we have that

$$\left\{ \begin{array}{l} \frac{\xi(z)}{a_t} - q \frac{|z|}{r_t} (1 + \delta_t) \leq \frac{\Phi_t(z)}{a_t} \leq \frac{\xi(z)}{a_t} - q \frac{|z|}{r_t} (1 - \delta_t) + \delta_t \\ \frac{\xi(z)}{a_t} - \frac{q}{1+\theta} \frac{|z|}{r_t} (1 + \delta_t) \leq \frac{\Phi_{t+\theta t}(z)}{a_t} \leq \frac{\xi(z)}{a_t} - \frac{q}{1+\theta} \frac{|z|}{r_t} (1 - \delta_t) + \delta_t \end{array} \right. .$$

Therefore, we can approximate 3.18 further by

$$\begin{aligned} & \text{Prob}\{Z_t = Z_{t(1+\theta_t)}\} \\ & \geq \sum_{|z| \leq r_t g_t} \int_{\eta_t}^{N_t} \text{Prob} \left\{ \begin{array}{ll} \frac{\xi(\bar{z})}{a_t} - q \frac{|\bar{z}|}{r_t} (1 - \delta_t) + \delta_t \leq y & \text{for } \bar{z} \neq z; \\ \frac{\xi(\bar{z})}{a_t} - \frac{q}{1+\theta_t} \frac{|\bar{z}|}{r_t} (1 - \delta_t) + \delta_t \leq y + \frac{q\theta_t}{1+\theta_t} (1 + \delta_t) & \text{for } \bar{z} \neq z; \\ \frac{\xi(z)}{a_t} - q \frac{|z|}{r_t} (1 + \delta_t) \in dy & \end{array} \right\} \end{aligned} \quad (3.19)$$

We now show that instead of having two conditions for all \bar{z} , one of the conditions is not necessary depending on whether $|z| < |\bar{z}|$ or $|z| > |\bar{z}|$. First of all, if $|\bar{z}| < |z|$ and we assume that

$$\frac{\xi(\bar{z})}{a_t} - q \frac{|\bar{z}|}{r_t} (1 - \delta_t) + \delta_t \leq y,$$

then we can deduce that

$$\frac{\xi(\bar{z})}{a_t} - \frac{q}{1+\theta_t} \frac{|\bar{z}|}{r_t} (1 - \delta_t) + \delta_t \leq y + \frac{q\theta_t}{1+\theta_t} \frac{|\bar{z}|}{r_t} (1 - \delta_t) \leq y + \frac{q\theta_t}{1+\theta_t} \frac{|z|}{r_t} (1 + \delta_t).$$

Conversely, if $|\bar{z}| > |z|$ and we assume that

$$\frac{\xi(\bar{z})}{a_t} - \frac{q}{1+\theta_t} \frac{|\bar{z}|}{r_t} (1 - \delta_t) + \delta_t \leq y + \frac{q\theta_t}{1+\theta_t} \frac{|z|}{r_t} (1 - \delta_t),$$

then it follows that

$$\frac{\xi(\bar{z})}{a_t} - q \frac{|\bar{z}|}{r_t} (1 - \delta_t) + \delta_t \leq y - \frac{q\theta_t}{1+\theta_t} \frac{|\bar{z}|}{r_t} (1 - \delta_t) + \frac{q\theta_t}{1+\theta_t} \frac{|z|}{r_t} (1 - \delta_t) \leq y.$$

Hence, we have found a lower bound which can be expressed using the independence of the ξ as

$$\begin{aligned} & \text{Prob}\{Z_t = Z_{t(1+\theta_t)}\} \\ & \geq \sum_{|z| \leq r_t g_t} \int_{\eta_t}^{N_t} \text{Prob}\left\{ \frac{\xi(z)}{a_t} - q \frac{|z|}{r_t} (1 + \delta) \in dy \right\} \prod_{|\bar{z}| < |z|} \text{Prob}\left\{ \frac{\xi(\bar{z})}{a_t} - q \frac{|\bar{z}|}{r_t} (1 - \delta_t) + \delta_t \leq y \right\} \\ & \quad \times \prod_{|\bar{z}| > |z|} \text{Prob}\left\{ \frac{\xi(\bar{z})}{a_t} - \frac{q}{1+\theta_t} \frac{|\bar{z}|}{r_t} (1 - \delta_t) + \delta_t \leq y + \frac{q\theta_t}{1+\theta_t} \frac{|z|}{r_t} (1 - \delta_t) \right\}. \end{aligned} \quad (3.20)$$

Again, we analyse the products separately. We can use that $\log(1 - x) \geq -x(1 + x)$ for

$0 < x < 1/2$ to see that

$$\begin{aligned}
& \prod_{|\bar{z}| < |z|} \text{Prob}\left\{\frac{\xi(\bar{z})}{a_t} - q \frac{|\bar{z}|}{r_t}(1 - \delta_t) + \delta_t \leq y\right\} \\
&= \exp\left\{\sum_{|\bar{z}| < |z|} \log\left(1 - a_t^{-\alpha}(y + q \frac{|\bar{z}|}{r_t}(1 - \delta_t) - \delta_t)^{-\alpha}\right)\right\} \\
&\geq (1 + o(1)) \exp\left\{-(1 + o(1)) \int_{|\bar{x}| < \frac{|z|}{r_t}} (y + q|\bar{x}|)^{-\alpha} d\bar{x}\right\},
\end{aligned}$$

where the approximation of the sum by the integral can be obtained as before, so that $o(1)$ is an error function that tends to 0 uniformly in $y \geq \eta_t$ and all $x \in \mathbb{Z}^d$. Similarly as in the upper bound, we can deal with the other products in (3.20) and approximate the sums by integrals to obtain

$$\begin{aligned}
& \text{Prob}\{Z_t = Z_{t(1+\theta_t)}\} \\
&\geq (1 + o(1)) \int_{|x| \leq g_t} \int_{\eta_t}^{N_t} \exp\left\{-(1 + o(1)) \int_{|\bar{x}| < |x|} (y + q|\bar{x}|)^{-\alpha} d\bar{x}\right\} \\
&\quad \times \exp\left\{-(1 + o(1)) \int_{|x| \leq |\bar{x}|} \left(y + \frac{q\theta_t}{1+\theta_t}|x| + \frac{q}{1+\theta_t}|\bar{x}|\right)^{-\alpha} d\bar{x}\right\} \frac{\alpha dx dy}{(y + q|x|)^{\alpha+1}}.
\end{aligned}$$

Note that we have obtained almost the same expression as for the upper bound. So in order to control the difference, we first estimate

$$\begin{aligned}
& \int_{|x| \geq g_t} \int_{\eta_t}^{N_t} \exp\left\{-(1 + o(1)) \int_{|\bar{x}| < |x|} (y + q|\bar{x}|)^{-\alpha} d\bar{x}\right\} \\
&\quad \times \exp\left\{-(1 + o(1)) \int_{|x| \leq |\bar{x}|} \left(y + \frac{q\theta_t}{1+\theta_t}|x| + \frac{q}{1+\theta_t}|\bar{x}|\right)^{-\alpha} d\bar{x}\right\} \frac{\alpha dx dy}{(y + q|x|)^{\alpha+1}} \\
&\leq \int_{|x| \geq g_t} \int_{\eta_t}^{N_t} \exp\left\{-(1 + o(1)) \int_{\bar{x} \in \mathbb{R}^d} (y + q|\bar{x}|)^{-\alpha} d\bar{x}\right\} \frac{\alpha dx dy}{(y + q|x|)^{\alpha+1}} \\
&= \frac{\alpha 2^d}{(d-1)!} \int_{\eta_t}^{N_t} \int_{r \geq g_t} e^{-(1+o(1))\vartheta y^{d-\alpha}} \frac{r^{d-1} dr dy}{(y + qr)^{\alpha+1}},
\end{aligned}$$

where we used the same simplification as in Proposition 3.1.4, in particular $\vartheta = \frac{2^d B(\alpha-d, d)}{q^d (d-1)!}$. In the same spirit, set $y + qr = y/v$, to get an upper bound on the previous display

$$\begin{aligned}
&\leq \frac{\alpha 2^d}{(d-1)! q^d} \int_0^\infty e^{-(1+o(1))\vartheta y^{d-\alpha}} y^{d-\alpha-1} \int_0^{N_t/(qg_t)} v^{\alpha-d} (1-v)^{d-1} dv dy \\
&\leq (1 + o(1)) \frac{\alpha}{B(\alpha-d, d)(\alpha-d)} \left(\frac{N_t}{qg_t}\right)^{\alpha-d+1}.
\end{aligned}$$

By first choosing N_t depending on θ_t and then g_t depending on θ_t and N_t , we can guarantee that this term is of order $o(\theta_t^{-d})$. Secondly, we can use a similar calculation

to estimate

$$\begin{aligned}
& \int_{x \in \mathbb{R}^d} \int_0^{\eta_t} \exp \left\{ - (1 + o(1)) \int_{|\bar{x}| < |x|} (y + q|\bar{x}|)^{-\alpha} d\bar{x} \right\} \\
& \quad \times \exp \left\{ - (1 + o(1)) \int_{|x| \leq |\bar{x}|} \left(y + \frac{q\theta_t}{1+\theta_t}|x| + \frac{q}{1+\theta_t}|\bar{x}| \right)^{-\alpha} d\bar{x} \right\} \frac{\alpha dx dy}{(y + q|x|)^{\alpha+1}} \\
& \leq \int_{x \in \mathbb{R}^d} \int_0^{\eta_t} e^{-(1+o(1)) \int_{\bar{x} \in \mathbb{R}^d} (y+q|\bar{x}|)^{-\alpha}} \frac{\alpha dx dy}{(y + q|x|)^{\alpha+1}} \\
& = \frac{\alpha 2^d}{(d-1)!q^d} \int_0^{\eta_t} e^{-(1+o(1))\vartheta y^{d-\alpha}} y^{d-\alpha-1} \int_0^1 v^{\alpha-d}(1-v)^{d-1} \\
& \leq \frac{\alpha 2^d B(\alpha-d+1, d)}{(d-1)!q^d} e^{-(1+o(1))\vartheta \eta_t^{d-\alpha}} \eta_t^{d-\alpha},
\end{aligned}$$

which is of order $o(\theta_t^{-d})$. Finally, we have to consider

$$\begin{aligned}
& \int_{x \in \mathbb{R}^d} \int_{N_t}^{\infty} \exp \left\{ - (1 + o(1)) \int_{|\bar{x}| < |x|} (y + q|\bar{x}|)^{-\alpha} d\bar{x} \right\} \\
& \quad \times \exp \left\{ - (1 + o(1)) \int_{|x| \leq |\bar{x}|} \left(y + \frac{q\theta_t}{1+\theta_t}|x| + \frac{q}{1+\theta_t}|\bar{x}| \right)^{-\alpha} d\bar{x} \right\} \frac{\alpha dx dy}{(y + q|x|)^{\alpha+1}} \\
& \leq \int_{x \in \mathbb{R}^d} \int_{N_t}^{\infty} \frac{\alpha dx dy}{(y + q|x|)^{\alpha+1}} \\
& \leq \frac{\alpha 2^d}{(d-1)!q^d} \int_{N_t}^{\infty} \int_0^1 v^{\alpha-d}(1-v)^{d-1} y^{\alpha-d-1} dv dy \leq C N_t^{\alpha-d},
\end{aligned}$$

for some constant $C > 0$, where again we can choose N_t such that this expression is of order $o(\theta_t^{-d})$.

Final step: Combining the upper and lower bound we have shown that

$$\begin{aligned}
& \text{Prob}\{Z_t = Z_{t(1+\theta_t)}\} \\
& = (1 + o(1)) \int_{y>0} \int_{x \in \mathbb{R}^d} \exp \left\{ - (1 + o(1)) \int_{|\bar{x}| < |x|} (y + q|\bar{x}|)^{-\alpha} d\bar{x} \right\} \\
& \quad \exp \left\{ - (1 + o(1)) \int_{|x| \leq |\bar{x}|} \left(y + \frac{q\theta_t}{1+\theta_t}|x| + \frac{q}{1+\theta_t}|\bar{x}| \right)^{-\alpha} d\bar{x} \right\} \frac{\alpha dx dy}{(y + q|x|)^{\alpha+1}} + o(\theta_t^{-d}).
\end{aligned}$$

Simplifying the integrals as in Proposition 3.1.4, we obtain that

$$\text{Prob}\{Z_t = Z_{t(1+\theta_t)}\} = (1 + o(1))I(\theta_t) + o(\theta_t^{-d}),$$

thus Proposition 3.1.5 completes the proof. \square

Remark 3.2.4. In fact, the proof of Proposition 3.2.1 even shows a slightly stronger statement. Namely, let $\gamma > 0$ and suppose ℓ_t is a function such that $\ell_t \rightarrow \infty$ as $t \rightarrow \infty$. Then for any $\varepsilon > 0$, there exists $T > 0$ such that for all $t \geq T$ and all $\ell_t \leq \theta \leq (\log t)^\gamma$,

we have that

$$(1 - \varepsilon) \frac{1}{d B(\alpha - d - 1)} \theta^{-d} \leq \text{Prob}\{Z_t = Z_{t+\theta t}\} \leq (1 + \varepsilon) \frac{1}{d B(\alpha - d - 1)} \theta^{-d}.$$

As we will see later on, the previous proposition suffices to give the first half of Theorem 1.2.3. For the second half, we also need to control the decay of correlations in the following way.

Lemma 3.2.5. *Let θ_t be a positive, nondecreasing function such that $\theta_t \rightarrow \infty$ as $t \rightarrow \infty$ and for some $\delta > 0$, $\theta_t \leq (\log t)^\delta$ for all $t > 0$. Then, for any $t > 0$ and $s \geq (1 + \theta_t)t$,*

$$\text{Prob}\{Z_t = Z_{t(1+\theta_t)} \neq Z_s = Z_{s(1+\theta_s)}\} \leq (1 + o(1)) \frac{1}{d^2 B(\alpha - d + 1, d)^2} \theta_t^{-d} \theta_s^{-d},$$

where $o(1)$ is an error term that vanishes as $t \rightarrow \infty$.

Proof. We use a similar notation as in the proof of Proposition 3.2.1. In particular, we will choose functions g_t, η_t, N_t depending on θ_t . Also, let $\delta_t = C \frac{\log \log t}{\log t}$, where C is the constant implied in the error bounds in Lemma 3.2.2. The first step is to show that if

$$\begin{aligned} & \text{Prob}\{Z_t = Z_{t(1+\theta_t)} \neq Z_s = Z_{s(1+\theta_s)}\} \\ &= \text{Prob} \left\{ \begin{array}{l} Z_t = Z_{t(1+\theta_t)} \neq Z_s = Z_{s(1+\theta_s)}; \\ \frac{\xi(Z_t)}{a_t} - q \frac{|Z_t|}{r_t} (1 - \delta_t) + \delta_t \in [\eta_t, N_t]; \\ \frac{\xi(Z_s)}{a_s} - q \frac{|Z_s|}{r_s} (1 - \delta_s) + \delta_s \in [\eta_s, N_s] \end{array} \right\} + \text{error}(t, s), \end{aligned} \quad (3.21)$$

then the error is of order $o(\theta_t^{-d} \theta_s^{-d})$. We will postpone this step to the subsequent Lemma 3.2.6, where we will see that for some constants $C_1, C_2 > 0$,

$$\text{error}(t, s) \leq C_1 (e^{-C_2 \eta_t^{d-\alpha}} + N_t^{d-\alpha}) (e^{-C_2 \eta_s^{d-\alpha}} + \theta_s^{-d} + N_s^{d-\alpha}) + C_1 \theta_t^{-d} (e^{-C_2 \eta_s^{d-\alpha}} + N_s^{d-\alpha}),$$

Taking $g_t = \theta_t^{q+3/2}$, and $N_t = g_t$ and $\eta_t = \theta_t^{-\beta'}$, where $0 < \beta' < \frac{1}{2}$, guarantees that the error is of order $o(\theta_t^{-d} \theta_s^{-d})$.

We can now concentrate on the probability on the right hand side of (3.21). Using Lemma 3.2.2, we find the following upper bound

$$\begin{aligned} & \text{Prob} \left\{ \begin{array}{l} Z_t = Z_{t(1+\theta_t)} \neq Z_s = Z_{s(1+\theta_s)}; \\ \frac{\xi(Z_t)}{a_t} - q \frac{|Z_t|}{r_t} (1 - \delta_t) + \delta_t \in [\eta_t, N_t]; \\ \frac{\xi(Z_s)}{a_s} - q \frac{|Z_s|}{r_s} (1 - \delta_s) + \delta_s \in [\eta_s, N_s] \end{array} \right\} \\ & \leq \sum_{z_1 \in \mathbb{Z}^d} \sum_{z_2 \in \mathbb{Z}^d \setminus \{z_1\}} \text{Prob} \left\{ \begin{array}{l} \Phi_{t(1+\theta_t)}(\bar{z}) \leq \Phi_{t(1+\theta_t)}(z_1) \quad \forall |\bar{z}| \leq r_t g_t; \bar{z} \neq z_1, z_2; \\ \Phi_{s(1+\theta_s)}(\bar{z}) \leq \Phi_{s(1+\theta_s)}(z_2) \quad \forall r_t g_t < |\bar{z}| \leq r_s g_s; \bar{z} \neq z_1, z_2; \\ \frac{\xi(Z_t)}{a_t} - q \frac{|Z_t|}{r_t} (1 - \delta_t) + \delta_t \in [\eta_t, N_t]; \\ \frac{\xi(Z_s)}{a_s} - q \frac{|Z_s|}{r_s} (1 - \delta_s) + \delta_s \in [\eta_s, N_s] \end{array} \right\}, \end{aligned}$$

which using the independence we can write as

$$\begin{aligned}
&\leq \sum_{(z_1, z_2), z_1 \neq z_2} \int_{\eta_t}^{N_t} \int_{\eta_s}^{N_s} \prod_{\substack{|\bar{z}| < g_t r_t \\ \bar{z} \neq z_1, z_2}} \text{Prob} \left\{ \frac{\xi(\bar{z})}{a_t} - q \frac{|\bar{z}|}{r_t} (1 + \delta_t) \leq y_1 + \frac{q\theta_t}{1+\theta_t} \frac{|z_1|}{r_t} (1 - \delta_t) \right\} \\
&\quad \times \prod_{\substack{g_t r_t < |\bar{z}| < g_s r_s \\ \bar{z} \neq z_1, z_2}} \text{Prob} \left\{ \frac{\xi(\bar{z})}{a_s} - q \frac{|\bar{z}|}{r_s} (1 + \delta_s) \leq y_2 + \frac{q\theta_s}{1+\theta_s} \frac{|z_2|}{r_s} (1 - \delta_s) \right\} \\
&\quad \times \text{Prob} \left\{ \frac{\xi(z_1)}{a_t} - q \frac{|z_1|}{r_t} (1 - \delta_t) + \delta_t \in dy_1 \right\} \text{Prob} \left\{ \frac{\xi(z_2)}{a_s} - q \frac{|z_2|}{r_s} (1 - \delta_s) + \delta_s \in dy_2 \right\}
\end{aligned}$$

As before, we can work out the probabilities, and approximate the sums by integrals to finally obtain $(1 + o(1))$ times

$$\begin{aligned}
&\int_{x_1 \in \mathbb{R}^d} \int_{y_1 \geq \eta_t}^{N_t} \exp \left\{ - (1 + o(1)) \int_{|\bar{x}| < g_t} \left(y_1 + \frac{q}{1+\theta_t} |\bar{x}| + \frac{q\theta_t}{1+\theta_t} |x_1| \right)^{-\alpha} d\bar{x} \right\} \frac{\alpha dx_1 dy_1}{(y_1 + q|x_1|)^{\alpha+1}} \\
&\times \int_{x_2 \in \mathbb{R}^d} \int_{y_2 \geq \eta_s}^{N_s} \exp \left\{ - (1 + o(1)) \int_{g_t r_t / r_s < |\bar{x}| < g_s} \left(y_2 + \frac{q}{1+\theta_s} |\bar{x}| + \frac{q\theta_s}{1+\theta_s} |x_2| \right)^{-\alpha} d\bar{x} \right\} \frac{\alpha dx_2 dy_2}{(y_2 + q|x_2|)^{\alpha+1}}.
\end{aligned} \tag{3.22}$$

In the remainder of the proof, we have to show that the first term is of order θ_t^{-d} , whereas the second is of order θ_s^{-d} . For the first factor, we use the same substitution as in Proposition 3.1.4 to turn the integral over \mathbb{R}^d into an integral over \mathbb{R}^+

$$\begin{aligned}
&\int_{|\bar{x}| < g_t} \left(y_1 + \frac{q}{1+\theta_t} |\bar{x}| + \frac{q\theta_t}{1+\theta_t} |x_1| \right)^{-\alpha} d\bar{x} = \frac{2^d}{(d-1)!} \int_{0 < r < g_t} \left(y_1 + \frac{q}{1+\theta_t} r + \frac{q\theta_t}{1+\theta_t} |x_1| \right)^{-\alpha} r^{d-1} dr \\
&\geq \frac{2^d}{(d-1)!} (1 + \theta_t)^d \int_{0 < r < g_t / (1+\theta_t)} \left(y_1 + qr + q|x_1| \right)^{-\alpha} r^{d-1} dr \\
&= \frac{2^d}{(d-1)!} (1 + \theta_t)^d \left\{ q^{-\alpha} (y_1 + q|x_1|)^{d-\alpha} B(\alpha - d, d) - \int_{r > \frac{g_t}{(1+\theta_t)}} \left(y_1 + qr + q|x_1| \right)^{-\alpha} r^{d-1} dr \right\}.
\end{aligned}$$

But now, if we consider

$$\begin{aligned}
(1 + \theta_t)^d \int_{r > g_t / (1+\theta_t)} \left(y_1 + qr + q|x_1| \right)^{-\alpha} r^{d-1} dr &\leq (1 + \theta_t)^d q^{-d} \int_{r > g_t / (1+\theta_t)} r^{d-\alpha-1} \\
&= q^{-\alpha} g_t^{d-\alpha} (1 + \theta_t)^\alpha,
\end{aligned}$$

we find that by our assumptions this expression tends to 0. Hence we can conclude that the first factor in (3.22) is bounded from above by the following expression, where $\vartheta = \frac{2^d B(\alpha-d, d)}{(d-1)! q^d}$ and we first reduce the integral over x_1 to a one-dimensional integral

and in the second step we use the substitution $y_1 + qr = y_1/v$.

$$\begin{aligned}
& (1 + o(1)) \int_{x_1 \in \mathbb{R}^d} \int_{y_1 \geq \eta_t}^{N_t} e^{-(1+o(1))\vartheta(1+\theta_t)^d(y+q|x_1|)^{d-\alpha}} \frac{\alpha \, dy_1 \, dx_1}{(y_1 + q|x_1|)^{\alpha+1}} \\
& (1 + o(1)) \frac{2^d}{(d-1)!} \int_{r>0} \int_{y_1>0} e^{-(1+o(1))\vartheta(1+\theta_t)^d(y+qr)^{d-\alpha}} \frac{\alpha \, r^{d-1} dy_1 \, dr}{(y_1 + qr)^{\alpha+1}} \\
& \leq (1 + o(1)) \frac{\alpha 2^d}{(d-1)! q^d} \int_{y \geq 0} \int_0^1 e^{-(1+o(1))\vartheta(1+\theta_t)^d y^{d-\alpha} v^{\alpha-d}} y^{\alpha-d-1} v^{\alpha-d} (1-v)^{d-1} dv \, dy \\
& = (1 + o(1)) (1 + \theta_t)^{-d} \frac{\alpha}{(\alpha-d)B(\alpha-d, d)} \int_0^1 (1-v)^{d-1} dv \\
& = (1 + o(1)) \theta_t^{-d} \frac{1}{dB(\alpha-d+1, d)}.
\end{aligned}$$

For the second factor in (3.22), we almost get the same expression, and it suffices to consider the following term and using similar substitutions to above, we can estimate uniformly in $y_2 \geq \eta_s$

$$\begin{aligned}
& \int_{|\bar{x}| < \frac{g_t r_t}{r_s}} (y_2 + \frac{q}{1+\theta_s} |\bar{x}| + q|x_2|)^{-\alpha} d\bar{x} \\
& = (1 + \theta_s)^d \frac{2^d}{(d-1)!} \int_{r < \frac{g_t r_t}{r_s(1+\theta_s)}} (y_2 + qr + q|x_2|)^{-\alpha} r^{d-1} dr \\
& = (1 + \theta_s)^d \frac{2^d}{q^d(d-1)!} (y_2 + q|x_2|)^{d-\alpha} \int_{\frac{y_2+q|x_2|}{y_2+q|x_2|+q\frac{g_t r_t}{r_s(1+\theta_s)}}}^1 u^{\alpha-d-1} (1-u)^{d-1} du \\
& \leq (1 + \theta_s)^d \frac{2^d}{q^d(d-1)!} (y_2 + q|x_2|)^{d-\alpha} \int_{1-\frac{qg_t r_t}{r_s(1+\theta_s)\eta_s}}^1 u^{\alpha-d-1} (1-u)^{d-1} du.
\end{aligned}$$

Now, we claim that the latter integral converges to 0. Indeed, using that $s/t \geq (1 + \theta_t)$ and recalling that $\eta_t = \theta_t^{-\beta'}$, where we can assume $0 < \beta' < 1$ and $g_t = \theta_t^{q+3/2}$, we obtain

$$\frac{g_t r_t}{r_s(1 + \theta_s)\eta_s} \leq \frac{g_t(\log t + \log(1 + \theta_t))^{q+1}}{(\log t)^{q+1} \theta_t^{q+2-\beta'}} \leq (1 + o(1)) \theta_t^{\beta' - \frac{1}{2}},$$

so that, since we chose $\beta' < \frac{1}{2}$, this term tends to 0. Now, we can simplify the second factor in (3.22) in the same way as the first one to show that it is of the required form. \square

To complete the proof of the previous lemma, we still have to show that the error term is small.

Lemma 3.2.6. *Using the notation from Lemma 3.2.5,*

$$\text{Prob} \left\{ \begin{array}{c} Z_t = Z_{t(1+\theta_t)} \neq Z_s = Z_{s(1+\theta_s)} \\ \text{and} \\ \left(\frac{\xi(Z_t)}{a_t} - q \frac{|Z_t|}{r_t} (1 - \delta_t) + \delta_t \notin [\eta_t, N_t]; \right. \\ \text{or} \\ \left. \frac{\xi(Z_s)}{a_s} - q \frac{|Z_s|}{r_s} (1 - \delta_s) + \delta_s \notin [\eta_s, N_s] \right) \end{array} \right\} \leq \text{error}(t, s),$$

where for some constants $C_1, C_2 > 0$ we have

$$\text{error}(t, s) \leq C_1 (e^{-C_2 \eta_t^{d-\alpha}} + N_t^{d-\alpha}) (e^{-C_2 \eta_s^{d-\alpha}} + \theta_s^{-d} + N_s^{d-\alpha}) + C_1 \theta_t^{-d} (e^{-C_2 \eta_s^{d-\alpha}} + N_s^{d-\alpha}).$$

Proof. We can bound the error from above by considering 8 different scenarios, which will correspond to the terms which we obtain when multiplying out the brackets in the statement of the lemma. The proof is based on calculations which are very similar to Lemma 3.2.3, in particular we will brief about corresponding parts. Throughout, we will denote by C, C_1, C_2 positive constants.

Before we indicate how to deal with each of those scenarios, recall from the proof of Lemma 3.2.5 that δ_t is chosen such that

$$\frac{\Phi_t(z)}{a_t} \leq \frac{\xi(z)}{a_t} - q \frac{|z|}{r_t} (1 - \delta_t).$$

Scenario 1 corresponds to the case that $\frac{\Phi_t}{a_t} \leq \eta_t$ and $\frac{\Phi_s}{a_s} \leq \eta_s$. Let g_t be defined as in the proof of Lemma 3.2.5. Then, we can estimate

$$\begin{aligned} \text{Prob} \left\{ \begin{array}{c} \frac{\xi(Z_t)}{a_t} - q \frac{|Z_t|}{r_t} (1 - \delta_t) + \delta_t \leq \eta_t \\ \frac{\xi(Z_s)}{a_s} - q \frac{|Z_s|}{r_s} (1 - \delta_s) + \delta_s \leq \eta_s \end{array} \right\} &\leq \text{Prob} \left\{ \frac{\Phi_t(Z_t)}{a_t} \leq \eta_t; \frac{\Phi_s(Z_s)}{a_s} \leq \eta_s \right\} \\ &\leq \prod_{\substack{|z| \leq r_t g_t \\ |z| \text{ odd}}} \text{Prob} \{ \Phi_t(z) \leq \eta_t a_t \} \prod_{\substack{|z| \leq r_s g_s \\ |z| \text{ even}}} \text{Prob} \{ \Phi_s(z) \leq \eta_s a_s \} \\ &= (1 + o(1)) \exp \left\{ - (1 + o(1)) \left[\int_{|x| \leq g_t/2} (\eta_t + 2q|x|)^{-\alpha} + \int_{|x| \leq g_s/2} (\eta_s + 2q|x|)^{-\alpha} \right] \right\}. \end{aligned}$$

Now the first integral is by the calculation in the proof of Lemma 3.2.3 of order $C \eta_t^{d-\alpha}$ and similarly, the second is of order $C \eta_s^{d-\alpha}$. Hence, we have shown that the probability of Scenario 1 decays like $C_1 e^{-C_2 \eta_t^{d-\alpha}} e^{-C_2 \eta_s^{d-\alpha}}$.

Scenario 2 corresponds to the case that $\frac{\xi(Z_t)}{a_t} - q \frac{|Z_t|}{r_t} (1 - \delta_t) + \delta_t \leq \eta_t$ and $Z_s = Z_{s(1+\theta_s)}$, where also $\frac{\xi(Z_t)}{a_t} - q \frac{|Z_t|}{r_t} (1 - \delta_t) + \delta_t \in [\eta_s, N_s]$. Now, the second part of the proof of Lemma 3.2.5 shows that in order to get the right asymptotics for the probability of the event $Z_s = Z_{s(1+\theta_s)}$ it suffices to consider all $z \in \mathbb{Z}^d$ such that $|z| > g_t r_t$ for a suitably chosen function g_t . However, the proof of Lemma 3.2.3 shows that for the right asymptotics of the first condition it suffices to take care of all z such that

$|z| \leq r_t g_t$ for any function $g_t \rightarrow \infty$. Hence, we can use independence to conclude that the probability of Scenario 2 is bounded by $C_1 e^{-C_2 \eta_t^{d-\alpha}} \theta_s^{-d}$.

Scenario 3 will be that $\frac{\xi(Z_t)}{a_t} - q \frac{|Z_t|}{r_t} (1 - \delta_t) + \delta_t \leq \eta_t$ and $\frac{\xi(Z_s)}{a_s} - q \frac{|Z_s|}{r_s} (1 - \delta_s) + \delta_s \geq N_s$. Then, the probability of this scenario can be bounded from above by

$$\begin{aligned} & \sum_{z_2 \in \mathbb{Z}^d} \text{Prob}\left\{\frac{\xi(z_2)}{a_s} - q \frac{|Z_s|}{r_s} (1 - \delta_s) + \delta_s \geq N_s; \Phi_t(z) \leq \eta_t \text{ for all } z \neq z_2 \text{ and } |z| \leq g_t r_t\right\} \\ & \leq \prod_{z \in \mathbb{Z}^d} \text{Prob}\{\Phi_t(z) \leq \eta_t a_t\} \sum_{z_2 \in \mathbb{Z}^d} \frac{\text{Prob}\left\{\frac{\xi(z_2)}{a_s} - q \frac{|z_2|}{r_s} (1 - \delta_s) + \delta_s \geq N_s\right\}}{\text{Prob}\{\Phi_t(z_2) \leq \eta_t a_t\}}. \end{aligned}$$

But, by Lemma 3.2.2,

$$\begin{aligned} \text{Prob}\{\Phi_t(z_2) \leq \eta_t a_t\} & \geq \text{Prob}\left\{\frac{\xi(z_2)}{a_t} - q \frac{|z_2|}{r_t} (1 - \delta_t) + \delta_t \leq \eta_t\right\} \\ & = 1 - a_t^{-\alpha} (\eta_t + q \frac{|z_2|}{r_t} (1 - \delta_t) + \delta_t)^{-\alpha} \geq 1 - a_t^{-\alpha} (\eta_t + \delta_t)^{-\alpha}, \end{aligned}$$

so that since $a_t \eta_t \rightarrow \infty$, this term tends to 1 uniformly in z_2 . In particular, the calculations in Lemma 3.2.3 show that the probability of Scenario 3 can be bounded by $C_1 e^{-C_2 \eta_t^{d-\alpha}} N_s^{d-\alpha}$.

Scenario 4 corresponds to the case that $\frac{\xi(Z_t)}{a_t} - q \frac{|Z_t|}{r_t} (1 - \delta_t) + \delta_t \geq N_t$ and $\frac{\xi(Z_s)}{a_s} - q \frac{|Z_s|}{r_s} (1 - \delta_s) + \delta_s \leq \eta_s$. This event can be bounded from above by

$$\begin{aligned} & \sum_{z_1 \in \mathbb{Z}^d} \text{Prob}\left\{\frac{\xi(Z_t)}{a_t} - q \frac{|Z_t|}{r_t} (1 - \delta_t) + \delta_t \geq a_t N_t; \Phi_s(z) \leq a_s \eta \text{ for all } z \neq z_1\right\} \\ & = \prod_{z \in \mathbb{Z}^d} \text{Prob}\{\Phi_s(z) \leq a_t \eta_s\} \sum_{z_1 \in \mathbb{Z}^d} \frac{\text{Prob}\left\{\frac{\xi(Z_t)}{a_t} - q \frac{|Z_t|}{r_t} (1 - \delta_t) + \delta_t \geq N_t\right\}}{\text{Prob}\{\Phi_s(z_1) \leq \eta_s a_s\}}. \end{aligned}$$

Again, as in the previous scenario the probability in the denominator tends to 1 uniformly in z_1 , so that we can deduce that the probability of this scenario decays like $C_1 N_t^{d-\alpha} e^{-C_2 \eta_s^{d-\alpha}}$.

Scenario 5 requires that $\frac{\xi(Z_t)}{a_t} - q \frac{|Z_t|}{r_t} (1 - \delta_t) + \delta_t \geq N_t$ and $Z_s = Z_{s(1+\theta_s)} \neq Z_t$, but also $\frac{\xi(Z_t)}{a_t} - q \frac{|Z_t|}{r_t} (1 - \delta_t) + \delta_t \in [\eta_s, N_s]$. As in the proof of Lemma 3.2.5, for some suitable

g_t the probability of this scenario can be bounded by

$$\begin{aligned}
& \sum_{(z_1, z_2), z_1 \neq z_2} \int_{\eta_s}^{N_s} \prod_{\substack{g_t r_t < |\bar{z}| < g_s r_s, \\ \bar{z} \neq z_1, z_2}} \text{Prob}\left\{\frac{\xi(\bar{z})}{a_t} - q \frac{|\bar{z}|}{r_t} (1 + \delta_t) - \delta_t \leq y_2 + \frac{q\theta_s}{1+\theta_s} \frac{|z_2|}{r_s} (1 - \delta_s)\right\} \\
& \times \text{Prob}\left\{\frac{\xi(z_1)}{a_t} - q \frac{|z_1|}{r_t} (1 - \delta_t) + \delta_t \geq N_t\right\} \text{Prob}\left\{\frac{\xi(z_2)}{a_s} - q \frac{|z_2|}{r_s} (1 - \delta_t) + \delta_t \in dy_2\right\} \\
& \leq \sum_{z_1} \int_{\eta_s}^{N_s} \prod_{\substack{g_t r_t < |\bar{z}| < g_s r_s, \\ \bar{z} \neq z_1}} \text{Prob}\left\{\frac{\xi(\bar{z})}{a_s} - q \frac{|\bar{z}|}{r_s} (1 + \delta_s) - \delta_s \leq y_2 + \frac{q\theta_s}{1+\theta_s} \frac{|z_2|}{r_s} (1 - \delta_s)\right\} \\
& \quad \times \text{Prob}\left\{\frac{\xi(z_2)}{a_s} - q \frac{|z_2|}{r_s} (1 - \delta_t) + \delta_t \in dy_2\right\} \\
& \quad \times \sum_{z_2 \neq z_1} \frac{\text{Prob}\left\{\frac{\xi(z_1)}{a_t} - q \frac{|z_1|}{r_t} (1 - \delta_t) + \delta_t \geq N_t\right\}}{\text{Prob}\left\{\frac{\xi(z_1)}{a_s} - q \frac{|z_1|}{r_s} (1 + \delta_s) - \delta_s \leq \eta_s\right\}}.
\end{aligned}$$

Now, as in Scenario 3, the probability in the denominator tends to 1 uniformly in z_1 and moreover,

$$\text{Prob}\left\{\frac{\xi(z_1)}{a_t} - q \frac{|z_1|}{r_t} (1 - \delta_t) + \delta_t \geq N_t\right\} \leq a_t^{-\alpha} N_t^{-\alpha} \rightarrow 0, \quad (3.23)$$

and we thus obtain the upper bound

$$\begin{aligned}
& (1 + o(1)) \sum_{z_1} \int_{\eta_s}^{N_s} \prod_{\substack{g_t r_t < |\bar{z}| < g_s r_s, \\ \bar{z} \neq z_1}} \text{Prob}\left\{\frac{\xi(\bar{z})}{a_s} - q \frac{|\bar{z}|}{r_s} (1 + \delta_s) - \delta_s \leq y_2 + \frac{q\theta_s}{1+\theta_s} \frac{|z_2|}{r_s} (1 - \delta_s)\right\} \\
& \quad \times \text{Prob}\left\{\frac{\xi(z_2)}{a_s} - q \frac{|z_2|}{r_s} (1 - \delta_t) + \delta_t \in dy_2\right\} \\
& \quad \times \sum_{z_2} \text{Prob}\left\{\frac{\xi(z_1)}{a_t} - q \frac{|z_1|}{r_t} (1 - \delta_t) + \delta_t \geq N_t\right\}.
\end{aligned}$$

The first expression can be simplified as in Lemma 3.2.5 to give a term $C\theta_s^{-d}$, while the second sum is of order $CN_t^{d-\alpha}$ by Lemma 3.2.3.

Scenario 6. In this case we assume that $\frac{\xi(Z_t)}{a_t} - q \frac{|Z_t|}{r_t} (1 - \delta_t) + \delta_t \geq N_t$ and $\frac{\xi(z_2)}{a_s} - q \frac{|z_2|}{r_s} (1 - \delta_s) + \delta_s \geq N_s$, where $Z_t \neq Z_s$. The probability of this event can be bounded by

$$\begin{aligned}
& \sum_{z_1 \neq z_2} \text{Prob}\left\{\frac{\xi(z_1)}{a_t} - q \frac{|z_1|}{r_t} (1 - \delta_t) + \delta_t \geq N_t\right\} \text{Prob}\left\{\frac{\xi(z_2)}{a_s} - q \frac{|z_2|}{r_s} (1 - \delta_s) + \delta_s \geq N_s\right\} \\
& \leq \sum_{z_1 \in \mathbb{Z}^d} \text{Prob}\left\{\frac{\xi(z_1)}{a_t} - q \frac{|z_1|}{r_t} (1 - \delta_t) + \delta_t \geq N_t\right\} \\
& \quad \times \left[\sum_{z_2 \in \mathbb{Z}^d} \text{Prob}\left\{\frac{\xi(z_2)}{a_s} - q \frac{|z_2|}{r_s} (1 - \delta_s) + \delta_s \geq N_s\right\} - \text{Prob}\left\{\frac{\xi(z_1)}{a_s} - q \frac{|z_1|}{r_s} (1 - \delta_s) + \delta_s \geq N_s\right\} \right].
\end{aligned}$$

Hence, the observation (3.23) together with the proof of Lemma 3.2.3 shows that this term is of order $CN_t^{d-\alpha}N_s^{d-\alpha}$.

Scenario 7. Here, we deal with the case hat $Z_t = Z_{t(1+\theta_t)}$ and $\frac{\xi(Z_t)}{a_t} - q \frac{|Z_t|}{r_t} (1 - \delta_t) + \delta_t \in [\eta_t, N_t]$, but $\frac{\xi(Z_s)}{a_s} - q \frac{|Z_s|}{r_s} (1 - \delta_s) + \delta_s \leq \eta_s$. We modify the proofs of Lemmas 3.2.3 and 3.2.5, for the same g_t , by specifying different conditions on $z \in \mathbb{Z}^d$ depending on whether $|z|$ is even or odd. In this way, we can find an upper bound on the probability of this scenario

$$\begin{aligned} & \sum_{z_1 : |z_1| \text{ even}} \int_{\eta_t}^{N_t} \prod_{\substack{|\bar{z}| < g_t r_t \\ \bar{z} \neq z_1, |\bar{z}| \text{ even}}} \text{Prob}\left\{\frac{\xi(\bar{z})}{a_t} - q \frac{|\bar{z}|}{r_t} (1 + \delta_t) \leq y_1 + \frac{q\theta_t}{1+\theta_t} \frac{|z_1|}{r_t} (1 - \delta_t)\right\} \\ & \times \text{Prob}\left\{\frac{\xi(z_1)}{a_t} - q \frac{|z_1|}{r_t} (1 - \delta_t) + \delta_t \in dy_1\right\} \prod_{\substack{|\bar{z}| < g_s r_s \\ |\bar{z}| \text{ odd}}} \text{Prob}\left\{\frac{\xi(\bar{z})}{a_s} - q \frac{|\bar{z}|}{r_s} (1 + \delta_s) \leq \eta_s\right\}. \end{aligned}$$

As before, we can work out the probabilities and approximate the sums by integrals, so that we get for the first term

$$\int_{x_1 \in \mathbb{R}^d} \int_{y_1 \geq \eta_t}^{N_t} \exp\left\{-(1+o(1)) \int_{|x| < g_t/2} (y_1 + \frac{2q}{1+\theta_t} |\bar{x}| + \frac{2q\theta_t}{1+\theta_t} |x_1|)^{-\alpha} d\bar{x}\right\} \frac{\alpha dx_1 dy_1}{(y_1 + 2q|x_1|)^{\alpha+1}}.$$

By the same calculation as in Lemma 3.2.5, this term is of order $C\theta_t^{-d}$, where, however, the constant is different. For the final expression in the upper bound, we obtain

$$\exp\left\{-(1+o(1)) \int_{|x| \leq g_s/2} (\eta_s + 2q|x|)^{-\alpha} dx\right\},$$

which can be bounded by $C_1 e^{-C_2 \eta_s^{d-\alpha}}$.

Scenario 8. The final case corresponds to $Z_t = Z_{t(1+\theta_t)}$ and $\frac{\xi(Z_t)}{a_t} - q \frac{|Z_t|}{r_t} (1 - \delta_t) + \delta_t \in [\eta_t, N_t]$, but $\frac{\xi(Z_s)}{a_s} - q \frac{|Z_s|}{r_s} (1 - \delta_s) + \delta_s \geq N_s$. This scenario can be controlled in the same way as Scenario 5 by simply interchanging the roles of t and s . This shows that the probability of this scenario is bounded by $C_1 c_t^{-d} N_s^{d-\alpha}$. \square

3.2.2 Almost sure asymptotics for the maximizer of the variational problem

In analogy with the residual lifetime function R for the process X_t , we can also define the residual lifetime function R^V for the maximizer $Z_t^{(1)}$ of the variational problem, by setting

$$R^V(t) = \sup\{s \geq 0 : Z_t^{(1)} = Z_{t+s}^{(1)}\}.$$

Using the moderate deviations principle, Proposition 3.2.1, developed in the previous section together with the Borel-Cantelli lemma, we aim to prove the following analogue of Theorem 1.2.3.

Proposition 3.2.7. *For any nondecreasing function $h : (0, \infty) \rightarrow (0, \infty)$ we have, almost surely,*

$$\limsup_{t \rightarrow \infty} \frac{R^V(t)}{th(t)} = \begin{cases} 0 & \text{if } \int_1^\infty \frac{dt}{th(t)^d} < \infty, \\ \infty & \text{if } \int_1^\infty \frac{dt}{th(t)^d} = \infty. \end{cases}$$

Proof of the first part of Proposition 3.2.7. Consider $h : (0, \infty) \rightarrow (0, \infty)$ such that $\int_1^\infty \frac{dt}{th(t)^d} < \infty$ which is equivalent to $\int_{t>1} h(\frac{1}{3}e^t)^{-d} dt < \infty$, so that

$$\sum_{n=1}^\infty h(\frac{1}{3}e^n)^{-d} < \infty. \quad (3.24)$$

This condition implies that $h(t) \rightarrow \infty$ as $t \rightarrow \infty$, for otherwise the monotonicity of h forces h to remain bounded, so that the above integrability statement cannot hold.

Additionally, without loss of generality, we can assume that there exists $\gamma > 1$ such that $\lim_{t \rightarrow \infty} h(t)(\log t)^{-\gamma} = 0$. Indeed, we can define $\tilde{h}(t) = \min\{(\log t)^{\frac{1}{2}(1+\gamma)}, h(t)\}$. Then, $t \mapsto t\tilde{h}(t)$ is increasing, since for $s < t$,

$$s\tilde{h}(s) \leq s \min\{(\log s)^{\frac{1}{2}(1+\gamma)}, h(s)\} \leq \min\{t(\log t)^{\frac{1}{2}(1+\gamma)}, th(t)\} = t\tilde{h}(t).$$

Also,

$$\begin{aligned} \int_{t>1} \frac{dt}{t\tilde{h}(t)^d} &= \int_{t>1} \max\left\{\frac{1}{th(t)^d}, \frac{1}{t(\log t)^{\frac{1}{2}(1+\gamma)}}\right\} dt \\ &\leq \int_{t>1} \frac{1}{th(t)^d} dt + \int_{t>1} \frac{1}{t(\log t)^{\frac{1}{2}(1+\gamma)}} dt < \infty. \end{aligned}$$

Therefore, the assumptions of the theorem apply to \tilde{h} and if we can prove the theorem in this case, we can deduce that since $\tilde{h} \leq h$,

$$\limsup_{t \rightarrow \infty} \frac{R(t)}{th(t)} \leq \limsup_{t \rightarrow \infty} \frac{R(t)}{t\tilde{h}(t)} = 0,$$

which justifies our assumption $\lim_{t \rightarrow \infty} h(t)(\log t)^{-\gamma} = 0$.

Fix $\varepsilon > 0$ and an increasing sequence $t_n \rightarrow \infty$. It suffices to show that almost surely,

$$\limsup_{n \rightarrow \infty} \frac{R(t_n)}{t_n h(t_n)} \leq \varepsilon.$$

Claim: Eventually for all n , if $\frac{R^V(t_n)}{t_n} > \varepsilon h(t_n)$ then $\frac{R^V(t)}{t} > \frac{1}{4}\varepsilon h(t_n)$ for all $t \in [t_n, 3t_n]$. Indeed, since by Lemma 3.1.8, Z_t never returns to a point that it has visited once, we know that if $\frac{R^V(t_n)}{t_n} > \varepsilon h(t_n)$, then Z_t does not jump on the interval $[t_n, t_n(1 + \varepsilon h(t_n))]$. In particular, R^V is affine with slope -1 on this interval, so that for $t \in$

$$[t_n, t_n \frac{(1+\varepsilon h(t_n))}{(1+\frac{1}{4}\varepsilon h(t_n))}],$$

$$\frac{R^V(t)}{t} = \frac{R^V(t_n) + t_n - t}{t} > \frac{(1 + \varepsilon h(t_n))t_n - t}{t} \geq \frac{\varepsilon h(t_n)}{4}.$$

To complete the proof of the claim, note that since the function h is unbounded, we have, eventually for all n ,

$$\frac{(1 + \varepsilon h(t_n))}{(1 + \frac{1}{4}\varepsilon h(t_n))} \geq 3.$$

Now, define $k(n) = \inf\{k : e^k \geq t_n\}$, so that in particular $t_n \leq e^{k(n)} < 3t_n$. Then by the claim, we can deduce that for n large enough

$$\frac{R^V(t_n)}{t_n} \geq \varepsilon h(t_n) \implies \frac{R^V(e^{k(n)})}{e^{k(n)}} \geq \frac{\varepsilon}{4} h(t_n) \geq \frac{\varepsilon}{12} h(\frac{1}{3}e^{k(n)}),$$

where we used in the last step that $\varphi(t) = th(t)$ is non-decreasing. This shows in particular that

$$\text{Prob}\left\{\frac{R^V(t_n)}{t_n} \geq \varepsilon h(t_n) \text{ infinitely often}\right\} \leq \text{Prob}\left\{\frac{R^V(e^n)}{e^n} \geq \frac{\varepsilon}{12} h(\frac{1}{3}e^n) \text{ infinitely often}\right\}.$$

It remains to show that the latter probability is 0. But since $h(t)(\log t)^\gamma \rightarrow 0$, we can invoke Proposition 3.2.1 to deduce that exists a constant \tilde{C} such that for all n large enough

$$\text{Prob}\left\{\frac{R^V(e^n)}{e^n} \geq \frac{\varepsilon}{4} h(\frac{1}{3}e^n)\right\} \leq \tilde{C} h(\frac{1}{3}e^n)^{-d}.$$

By the integrability assumption on h , see (3.24), these probabilities are summable so that the Borel-Cantelli lemma completes the proof. \square

For the second part of Proposition 3.2.7, we need to prove a lower bound on the limit superior, so our strategy is to use the fine control over the decay of correlations that we developed in the previous section and combine it with the Kochen-Stone lemma.

Proof of second part of Proposition 3.2.7. Let $h : (0, \infty) \rightarrow (0, \infty)$ be such that $\int_1^\infty \frac{dt}{th(t)^d} = \infty$. Then, we can deduce that

$$\sum_{n=1}^{\infty} h(e^n)^{-d} = \infty. \tag{3.25}$$

Without loss of generality, we can assume that $h(t) \rightarrow \infty$ and also that $h(t) \leq (\log t)^2$ for all $t \geq 0$. Namely, take $\tilde{h}(t) = \min\{(\log t)^2, h(t)\}$. Then,

$$\int_{t>1} \frac{dt}{t\tilde{h}(t)^d} \geq \int_{t>1} \frac{dt}{th(t)^d} = \infty, .$$

Clearly $\tilde{h}(t) \leq (\log t)^2$ and so if we can prove the theorem under this extra assumption,

we can deduce that there exists a sequence $t_n \rightarrow \infty$ such that

$$\lim_{n \rightarrow \infty} \frac{R(t_n)}{t_n \tilde{h}(t_n)} = \infty.$$

Now, if there exists a subsequence t_{n_k} such that $h(t_{n_k}) > (\log t_{n_k})^2$, then $\tilde{h}(t_{n_k}) = \log(t_{n_k})^2$ and it follows that

$$\lim_{k \rightarrow \infty} \frac{R(t_{n_k})}{t_{n_k} (\log(t_{n_k}))^2} = \infty,$$

which contradicts the first part of the theorem, as $\sum_{k \geq 1} \frac{1}{n^2}$ is summable. Hence, we know that eventually for all n large enough, $\tilde{h}(t_n) = h(t_n)$ and we can deduce that

$$\lim_{n \rightarrow \infty} \frac{R(t_n)}{t_n h(t_n)} = \lim_{n \rightarrow \infty} \frac{R(t_n)}{t_n \tilde{h}(t_n)} = \infty,$$

so that the theorem also holds for h .

For $\kappa > 0$, define the event

$$E_n = \left\{ \frac{R(e^n)}{e^n} \geq \kappa h(e^n) \right\}.$$

By Proposition 3.2.1, we know that

$$\text{Prob}(E_n) = \frac{1}{d B(\alpha - d + 1, d)} (1 + o(1)) \kappa^{-d} h(e^n)^{-d},$$

so that by the integral condition, we know that $\sum_{k=1} \text{Prob}(E_n) = \infty$. We want to use the result from the previous section to show that E_n and E_m are asymptotically independent for $m \neq n$ to deduce that the probability that the events E_n occur infinitely often is equal to 1. More precisely, by the Kochen-Stone lemma, see for instance [FG97], we find that

$$\text{Prob}\{E_n \text{ infinitely often}\} \geq \limsup_{k \rightarrow \infty} \frac{(\sum_{n=1}^k \text{Prob}(E_n))^2}{\sum_{n=1}^k \sum_{m=1}^k \text{Prob}(E_m \cap E_n)}. \quad (3.26)$$

Fix $\varepsilon > 0$. By Proposition 3.2.1 and Remark 3.2.4 we can deduce that we can choose N large enough such that for all $t \geq N$ and all $(\log t)^{1/d} \wedge h(t) \leq \theta \leq (\log t)^6$, we have that

$$(1 - \varepsilon) \frac{1}{d B(\alpha - d + 1, d)} \theta^{-d} \leq \text{Prob}\{Z_t = Z_{t+\theta t}\} \leq (1 + \varepsilon) \frac{1}{d B(\alpha - d + 1, d)} \theta^{-d}. \quad (3.27)$$

Also, by Lemma 3.2.5, we know that we can assume N is large enough such that such

that for all $n \geq N$ and $m \geq n + \log(1 + \kappa h(e^n))$, we have that

$$\begin{aligned} \text{Prob}\{Z_{e^n} = Z_{e^{n(1+\kappa h(e^n))}} \neq Z_{e^m} = Z_{e^{m(1+\kappa h(e^m))}}\} \\ \leq (1 + \varepsilon) \left(\frac{1}{dB(\alpha-d+1, 1)} \right)^2 \kappa^{-2d} h(e^n)^{-d} h(e^m)^{-d} \quad (3.28) \\ \leq \frac{1+\varepsilon}{1-\varepsilon} \text{Prob}(E_n) \text{Prob}(E_m). \end{aligned}$$

Note that by Lemma 3.1.8, we know that Z_t never returns to the same point, therefore we have

$$\begin{aligned} \text{Prob}(E_n \cap E_m) \\ = \text{Prob}\{Z_{e^n} = Z_{e^{m(1+\kappa h(e^n))}}\} + \text{Prob}\{Z_{e^n} = Z_{e^{n(1+\kappa h(e^n))}} \neq Z_{e^m} = Z_{e^{m(1+\kappa h(e^m))}}\} \end{aligned}$$

In particular, notice that the second probability is zero if $n \leq m \leq n(1 + \kappa h(e^n))$. Hence, we can estimate for $n > N$ and for k large enough, using (3.27) and (3.28),

$$\begin{aligned} \sum_{m=n}^k \text{Prob}(E_n \cap E_m) \\ \leq \sum_{m=n}^{n+2 \log n} \text{Prob}\{Z_{e^n} = Z_{e^{m(1+\kappa h(e^n))}}\} + \sum_{m=n+2 \log n}^{\ell} \text{Prob}\{Z_{e^{m_{n-2}}} = Z_{e^{m(1+\kappa h(e^m))}}\} \\ + \frac{1+\varepsilon}{1-\varepsilon} \sum_{m \geq n + \log(1+\kappa h(e^n))}^{\ell} \text{Prob}(E_n) \text{Prob}(E_m) \\ \leq \tilde{C} \text{Prob}(E_n) \sum_{m=n}^{\ell} e^{d(n-m)} + \tilde{C} n^{-2d} \sum_{m=n}^{\ell} \text{Prob}(E_m) + \frac{1+\varepsilon}{1-\varepsilon} \sum_{m=n}^{\ell} \text{Prob}(E_n) \text{Prob}(E_m), \end{aligned}$$

where \tilde{C} is some suitable constant. Finally, in order to bound the right hand side of (3.26), we can estimate for $k > N$

$$\begin{aligned} \sum_{n=1}^k \sum_{m=1}^k \text{Prob}(E_n \cap E_m) &\leq 2N \sum_{n=1}^k \text{Prob}(E_n) + \sum_{n=N}^k \sum_{m=N}^k \text{Prob}(E_n \cap E_m) \\ &\leq 2 \sum_{n=1}^k \left(N + \sum_{m=1}^k m^{-2d} + \sum_{m=n}^k \tilde{C} e^{d(n-m)} \right) \text{Prob}(E_n) \\ &\quad + 2 \frac{1+\varepsilon}{1-\varepsilon} \sum_{n=N}^k \sum_{m=n}^k \text{Prob}(E_n) \text{Prob}(E_m) \\ &\leq C' \sum_{n=1}^k \text{Prob}(E_n) + \frac{1+\varepsilon}{1-\varepsilon} \sum_{n=1}^k \sum_{k=1}^k \text{Prob}(E_n) \text{Prob}(E_m), \end{aligned}$$

where $C' > 0$. Therefore, we can conclude from the Kochen-Stone lemma (3.26) that

$$\text{Prob}\{E_n \text{ infinitely often} \} \geq \frac{1-\varepsilon}{1+\varepsilon}.$$

Since $\varepsilon > 0$ was arbitrary, we deduce that almost surely,

$$\limsup_{t \rightarrow \infty} \frac{R(t)}{th(t)} \geq \kappa,$$

and thus, since κ was arbitrary, the second statement of Proposition 3.2.7 follows. \square

3.2.3 Almost sure asymptotics for the maximizer of the solution profile

In this section, we will prove Theorem 1.2.3. Thus, we have to transfer the almost sure ageing result of Proposition 3.2.7, which was formulated on the level of the variational problem, to the residual lifetime function of the maximizer X_t of the profile v . The underlying idea is that most of the time X_t and the maximizer of the variational problem Z_t agree and we only have to control the length of the intervals when they can disagree. The latter scenario corresponds to those times during which the processes relocate to another point. Therefore, our strategy is to look at the jump times and show that both processes jump almost at the same time.

The period when the maximizers relocates correspond exactly to those times when $Z_t^{(1)}$ and $Z_t^{(2)}$ produce a comparable value of Φ . With this in mind, define for $\lambda_t = (\log t)^{-\beta}$ with $\beta > 1 + \frac{1}{\alpha-d}$, the set of exceptional transition times

$$\mathcal{E} := \mathcal{E}(\beta) = \{t > t_0 : \Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(2)}) \leq \frac{1}{2}a_t\lambda_t\}, \quad (3.29)$$

where t_0 is chosen sufficiently large and to avoid trivialities such that $t_0 \neq \inf \mathcal{E}$. By [KLMS09, Lemma 3.4] we can choose t_0 large enough such that for all $t > t_0$,

$$\Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(3)}) > a_t\lambda_t. \quad (3.30)$$

Lemma 3.2.8. *The process $(Z_t^{(1)})_{t \geq t_0}$ jumps only at times contained in the interval \mathcal{E} . Moreover, each connected component of \mathcal{E} contains exactly one such jump time.*

Proof. The first part of the statement is trivial, since at each jump time $\tau \geq t_0$ of $Z_t^{(1)}$ we have that $\Phi_\tau(Z_\tau^{(1)}) = \Phi_\tau(Z_\tau^{(2)})$ so that $\tau \in \mathcal{E}$. For the second statement, let $[b^-, b^+]$ be a connected component of \mathcal{E} , then

$$\Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(2)}) = \frac{1}{2}a_t\lambda_t,$$

for $t = b^-, b^+$ (here we used that $b^- \geq \inf \mathcal{E} \neq t_0$). Now, since $t \mapsto \Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(2)})$ is never constant, if $Z_{b^-}^{(1)} = Z_{b^+}^{(1)}$ then by Lemma 3.1.10 there is $t \in (b^-, b^+)$ such that $t \notin \mathcal{E}$ contradicting the definition of $[b^-, b^+]$ as a connected component. Thus, we can conclude that $Z_t^{(1)}$ jumps at least once in $[b^-, b^+]$. Finally, the fact that, by Lemma 3.1.8, $Z_t^{(1)}$ never returns to the same point combined with (3.30) guarantees that $Z_t^{(1)}$ only jumps once in $[b^-, b^+]$ (namely from $Z_{b^-}^{(1)}$ to $Z_{b^+}^{(1)}$). \square

We now collect some basic properties of two consecutive jump times of Z_t , to prove

first a lower bound on their difference and secondly a lower bound on the difference in potential of the first and second maximizer. Denote by (τ_n) the jump times of the maximizer process $(Z_t^{(1)})_{t \geq t_0}$ in increasing order.

Lemma 3.2.9. (i) Fix $\beta > 1 + \frac{1}{\alpha-d}$, then, almost surely, eventually for all n ,

$$(\xi(Z_{\tau_n}^{(1)}) - \xi(Z_{\tau_n}^{(2)})) \left(\frac{\tau_{n+1} - \tau_n}{\tau_n} \right) \geq a_{\tau_n} (\log \tau_n)^{-\beta};$$

(ii) Fix $\gamma > 1 + \frac{2}{\alpha-d}$, then, almost surely, eventually for all n ,

$$\frac{\tau_{n+1} - \tau_n}{\tau_n} \geq (\log \tau_n)^{-\gamma}.$$

(iii) Fix $\delta > 1 + \frac{1}{\alpha-d} + \frac{1}{d}$, then, almost surely, eventually for all n

$$\xi(Z_{\tau_n}^{(1)}) - \xi(Z_{\tau_n}^{(2)}) \geq a_{\tau_n} (\log \tau_n)^{-\delta}.$$

Proof. (i) By Lemma 3.1.6 and Remark 3.1.7 we find that

$$\begin{aligned} \Phi_{\tau_{n+1}}(Z_{\tau_n}^{(1)}) - \Phi_{\tau_{n+1}}(Z_{\tau_n}^{(2)}) &= (\xi(Z_{\tau_n}^{(1)}) - \xi(Z_{\tau_n}^{(2)})) \left(\frac{\tau_{n+1} - \tau_n}{\tau_{n+1}} \right) \\ &\leq (\xi(Z_{\tau_n}^{(1)}) - \xi(Z_{\tau_n}^{(2)})) \left(\frac{\tau_{n+1} - \tau_n}{\tau_n} \right). \end{aligned} \quad (3.31)$$

Now, we can estimate the difference on the left-hand side from below by using that $Z_{\tau_n}^{(2)}$ cannot produce more than the third largest value of Φ at time τ_{n+1} . Indeed, Lemma 3.1.8 ensures that $Z_{\tau_n}^{(1)}$ never visits the same point again, so that $Z_{\tau_n}^{(2)} = Z_{\tau_{n-1}}^{(1)} \neq Z_{\tau_{n+1}}^{(i)}$ for $i = 1, 2$ since $Z_{\tau_{n+1}}^{(2)} = Z_{\tau_n}^{(1)}$. Hence, using [KLMS09, Proposition 3.4] for the second inequality,

$$\begin{aligned} \Phi_{\tau_{n+1}}(Z_{\tau_n}^{(2)}) &\leq \Phi_{\tau_{n+1}}(Z_{\tau_{n+1}}^{(3)}) \leq \Phi_{\tau_{n+1}}(Z_{\tau_{n+1}}^{(1)}) - a_{\tau_{n+1}} (\log \tau_{n+1})^{-\beta} \\ &\leq \Phi_{\tau_{n+1}}(Z_{\tau_n}^{(1)}) - a_{\tau_n} (\log \tau_n)^{-\beta}, \end{aligned}$$

where in the last step we again used that $Z_{\tau_n}^{(1)} = Z_{\tau_{n+1}}^{(2)}$ and that $t \mapsto a_t (\log t)^{-\beta}$ is increasing for all sufficiently large t . Substituting this inequality into (3.31) completes the proof of part (i).

(ii) By the first part, we need to get an upper bound on $\xi(Z_{\tau_n}^{(1)})$. Therefore, our first claim is that for any $\delta > \frac{1}{\alpha-d}$, and all t sufficiently large

$$\xi(Z_t^{(1)}) \leq a_t (\log t)^\delta. \quad (3.32)$$

Indeed, by [HMS08, Lemma 3.5], for $\varepsilon = \frac{1}{3}(\delta - \frac{1}{\alpha-d})$ we have eventually for all r ,

$$\max_{|z| \leq r} \xi(z) \leq r^{\frac{d}{\alpha}} (\log r)^{\frac{1}{\alpha} + \varepsilon}.$$

Moreover, by [KLMS09, Lemma 3.2], for $\varepsilon' = \frac{\alpha}{d}(\frac{1}{3}(\delta - \frac{1}{\alpha-d}))$, we find that $|Z_t^{(1)}| < r_t(\log t)^{\frac{1}{\alpha-d}+\varepsilon'}$ and therefore using that $r_t^{\frac{d}{\alpha}} = a_t$ and that $\log r_t = (q+1)\log t(1+o(1))$,

$$\begin{aligned}\xi(Z_t^{(1)}) &\leq a_t(\log t)^{(\frac{1}{\alpha-d}+\varepsilon')\frac{d}{\alpha}}(\log(r_t(\log t)^{\frac{1}{\alpha-d}+\varepsilon'}))^{\frac{1}{\alpha}+\varepsilon} \\ &= a_t(q+1)^{\frac{1}{\alpha}+\varepsilon}(\log t)^{\frac{1}{\alpha-d}+\varepsilon'\frac{d}{\alpha}+\varepsilon}(1+o(1)) < a_t(\log t)^\delta,\end{aligned}$$

eventually for all t sufficiently large.

Now, if we combine part (i) for $\beta = \frac{1}{2}(\gamma+1) > 1 + \frac{1}{\alpha-d}$ with bound (3.32) for $\delta = \frac{1}{2}(\gamma-1) > \frac{1}{\alpha-d}$, we have that

$$\frac{\tau_{n+1} - \tau_n}{\tau_n} \geq \frac{a_{\tau_n}(\log \tau_n)^{-\beta}}{\xi(Z_{\tau_n}^{(1)})} \geq \frac{a_{\tau_n}(\log \tau_n)^{-\beta}}{a_{\tau_n}(\log \tau_n)^\delta} = (\log \tau_n)^{-(\delta+\beta)} = (\log \tau_n)^{-\gamma},$$

which completes the proof of the lemma.

(iii) Note that for any $\delta' > \frac{1}{d}$, Proposition 3.2.7, shows that eventually for all n ,

$$\frac{\tau_{n+1} - \tau_n}{\tau_n} \leq (\log \tau_n)^{\delta'}.$$

This observation together with part (i), immediately implies the statement of part (iii). \square

A similar statement to Lemma 3.2.8 also holds for the process $X_t^{(1)} = \operatorname{argmax}\{u(t, z) : z \in \mathbb{Z}^d\}$. Fix $0 < \varepsilon < \frac{1}{3}$, then by [KLMS09, Proposition 5.3] we can assume additionally that t_0 in the definition (3.29) of \mathcal{E} is chosen large enough such that for all $t > t_0$

$$\left[U(t)^{-1} \sum_{z \in \mathbb{Z}^d} u(t, z) \right] \mathbb{I}\{\Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(2)}) \geq \frac{1}{2}a_t\lambda_t\} < \varepsilon. \quad (3.33)$$

Furthermore, by the ‘two cities theorem’ [KLMS09, Theorem 1.1], we may assume that for all $t \geq t_0$,

$$\frac{u(t, Z_t^{(1)}) + u(t, Z_t^{(2)})}{U(t)} > 1 - \varepsilon. \quad (3.34)$$

Lemma 3.2.10. *The process $(X_t)_{t \geq t_0}$ only jumps at times contained in \mathcal{E} and each connected component of \mathcal{E} contains exactly one such jump time. Furthermore, $(X_t)_{t \geq t_0}$ never returns to the same point in \mathbb{Z}^d .*

Proof. By (3.33), we find that for any $t \in [t_0, \infty) \setminus \mathcal{E}$

$$u(t, Z_t^{(1)}) > (1 - \varepsilon)U(t) > \frac{1}{2} \sum_{z \in \mathbb{Z}^d} u(t, z),$$

so that $u(t, Z_t^{(1)}) > u(t, z)$ for any $z \neq Z_t^{(1)}$. This implies that $X_t^{(1)} = Z_t^{(1)}$ so that in particular X_t jumps only at times in \mathcal{E} .

Now, let $[b^-, b^+]$ be a connected component of \mathcal{E} . Note that the proof of Lemma 3.2.8 shows that for all $t \in [b^-, b^+]$, the set $\{Z_t^{(1)}, Z_t^{(2)}\}$ consists of exactly two points, $z^{(1)} := Z_{b^+}^{(1)}$ and $z^{(2)} := Z_{b^+}^{(2)} = Z_{b^-}^{(1)}$. Hence, by (3.33) we find that

$$\begin{aligned} u(b^-, z^{(2)}) &= u(b^-, Z_{b^-}^{(1)}) > (1 - \varepsilon)U(b^-) \\ u(b^+, z^{(1)}) &= u(b^+, Z_{b^+}^{(1)}) > (1 - \varepsilon)U(b^+), \end{aligned} \quad (3.35)$$

so that $X_{b^-}^{(1)} = z^{(2)}$ and $X_{b^+}^{(1)} = z^{(1)}$. Also, since $\varepsilon < \frac{1}{3}$, the two-point localisation (3.34) implies that

$$\{X_t^{(1)} : t \in [b^-, b^+]\} = \{z^{(1)}, z^{(2)}\}.$$

Hence, we have to show that $X_t^{(1)}$ jumps only once (from $z^{(2)}$ to $z^{(1)}$) in the interval $[b^-, b^+]$.

Define the function

$$g(t) = \frac{u(t, z^{(1)})}{u(t, z^{(2)})}.$$

Then, note that since u solves the heat equation, for $z = z^{(1)}$ or $z^{(2)}$

$$\partial_t u(t, z) = \Delta u(t, z) + \xi(z)u(t, z) = \sum_{y \sim z} (u(t, y) - u(t, z)) + \xi(z)u(t, z).$$

Furthermore, [KLMS09, Lemmas 2.2, 3.2] tells us that $z^{(1)} \not\sim z^{(2)}$ so that we can use (3.34) to estimate

$$\begin{aligned} (-2d + \xi(z))u(t, z) &< \partial_t u(t, z) < 2d\varepsilon U(t) - 2du(t, z) + \xi(z)u(t, z) \\ &< 2d\frac{\varepsilon}{1 - \varepsilon}(u(t, z^{(1)}) + u(t, z^{(2)})) + (\xi(z) - 2d)u(t, z). \end{aligned}$$

Therefore, if we calculate the derivative of g we obtain

$$\begin{aligned} g'(t) &= \frac{\partial_t u(t, z^{(1)})u(t, z^{(2)}) - u(t, z^{(1)})\partial_t u(t, z^{(2)})}{u(t, z^{(2)})^2} \\ &> \frac{1}{u(t, z^{(2)})^2} \left[(\xi(z^{(1)}) - \xi(z^{(2)}) - 2d\frac{\varepsilon}{1 - \varepsilon})u(t, z^{(1)})u(t, z^{(2)}) - 2d\frac{\varepsilon}{1 - \varepsilon}u(t, z^{(1)})^2 \right] \\ &= g(t) \left(\xi(z^{(1)}) - \xi(z^{(2)}) - 2d\frac{\varepsilon}{1 - \varepsilon}(1 + g(t)) \right) \end{aligned}$$

Now, since $z^{(1)} = Z_{b^+}^{(1)}$ and $z^{(2)} = Z_{b^-}^{(1)}$, Lemma 3.2.9 shows (again assuming that t_0 is large enough) that, for any $\delta > 1 + \frac{1}{\alpha - d} + \frac{1}{d}$, if τ is the jump time of $Z^{(1)}$ in the interval $[b^-, b^+]$, then

$$\xi(z^{(1)}) - \xi(z^{(2)}) \geq a_\tau (\log \tau)^{-\delta}.$$

Hence, we can deduce that if there exists t' such that $g(t') = 1$, then $g'(t') > 0$. Thus, the continuity of u implies that there can be at most one such t' and $g(t) < 1$ if $t < t'$ and $g(t) > 1$ if $t > t'$. However, we can deduce from (3.35) and the continuity of u that there exists $t' \in [b^-, b^+]$ such that $g(t') = 1$, therefore it has to be unique and

$u(t, z^{(1)}) < u(t, z^{(2)})$ if $t < t'$ and $u(t, z^{(1)}) > u(t, z^{(2)})$ if $t > t'$. Thus, we can see that $X_t^{(1)}$ jumps exactly once in the interval $[b^-, b^+]$.

To complete the proof, we just have to notice that by Lemma 3.1.8, we know that $t \mapsto \xi(Z_t^{(1)})$ is increasing on (t_0, ∞) . Therefore since $Z^{(1)}$ and $X^{(1)}$ agree on $[t_0, \infty) \setminus \mathcal{E}$ and since in any connected component $[b^-, b^+]$ of \mathcal{E} , $X_t^{(1)}$ jumps once from $Z_{b^-}^{(1)}$ to $Z_{b^+}^{(1)}$, we conclude that $t \mapsto \xi(X_t^{(1)})$ is increasing, so that once X_t has jumped, it never returns to a point that it has visited previously. \square

In order to be able to deduce the asymptotics of the jump times of X_t from those of Z_t , we find bounds for the length of a connected component of \mathcal{E} .

Lemma 3.2.11. *Suppose in the definition (3.29) we choose $\beta > 1 + \frac{q+2}{d} + \frac{1}{\alpha-d}$. Then, for any $0 < \varepsilon < \frac{1}{2}(\beta - (1 + \frac{q+2}{d} + \frac{1}{\alpha-d}))$, almost surely for any connected component $[b^-, b^+]$ of \mathcal{E} with b^- large enough, we find that*

$$\frac{b^+ - b^-}{\tau} \leq (\log \tau)^{-\varepsilon},$$

where τ is the jump time of the process $Z_t^{(1)}$ in the interval $[b^-, b^+]$.

Proof. We start by expressing the distances $b^+ - \tau$ and $\tau - b^-$ in terms of the potential values at the sites $Z_\tau^{(1)}$ and $Z_\tau^{(2)}$. As we have seen in the proof of Lemma 3.2.8, $Z_\tau^{(i)} = Z_{b^\pm}^{(i)}$ for $i = 1, 2$. Hence, we obtain that

$$\Phi_{b^+}(Z_\tau^{(1)}) - \Phi_{b^+}(Z_\tau^{(2)}) = \Phi_{b^+}(Z_{b^+}^{(1)}) - \Phi_{b^+}(Z_{b^+}^{(2)}) = \frac{1}{2}a_{b^+}\lambda_{b^+}.$$

Moreover, by Lemma 3.1.6 we get that

$$\Phi_{b^+}(Z_\tau^{(1)}) - \Phi_{b^+}(Z_\tau^{(2)}) = (\xi(Z_\tau^{(1)}) - \xi(Z_\tau^{(2)}))(1 - \frac{\tau}{b^+}).$$

Combining the previous two displayed equations and rearranging yields

$$b^+ - \tau = \frac{\frac{1}{2}b^+a_{b^+}\lambda_{b^+}}{\xi(Z_\tau^{(1)}) - \xi(Z_\tau^{(2)})}. \quad (3.36)$$

Similarly, we know that $Z_{b^-}^{(1)} = Z_\tau^{(2)}$ and $Z_{b^-}^{(2)} = Z_\tau^{(1)}$ and can therefore deduce in the same way that

$$\tau - b^- = \frac{\frac{1}{2}b^-a_{b^-}\lambda_{b^-}}{\xi(Z_\tau^{(1)}) - \xi(Z_\tau^{(2)})}. \quad (3.37)$$

Now, define τ^+ as the next jump of $Z_t^{(1)}$ after τ , then $b^+ \leq \tau^+$ and we can use (3.36) and (3.37) to estimate

$$\begin{aligned} \frac{b^+ - b^-}{\tau} &= \frac{b^+ - \tau}{\tau} + \frac{\tau - b^-}{\tau} = \frac{1}{2} \frac{1}{\xi(Z_\tau^{(1)}) - \xi(Z_\tau^{(2)})} \left(a_{b^+}\lambda_{b^+} \frac{b^+}{\tau} + a_{b^-}\lambda_{b^-} \frac{b^-}{\tau} \right) \\ &\leq \frac{1}{2} \frac{1}{\xi(Z_\tau^{(1)}) - \xi(Z_\tau^{(2)})} \left(a_{b^+}\lambda_{b^+} \frac{\tau^+}{\tau} + a_\tau\lambda_\tau \right), \end{aligned} \quad (3.38)$$

where we used in the last step that $\beta^- \leq \tau$ and that $t \mapsto a_t(\log t)^{-\beta} = \frac{t^q}{(\log t)^{q+\beta}}$ is increasing for all t large enough. Next, by the definition of a_t and λ_t , we obtain that

$$a_{b^+} \lambda_{b^+} = \frac{(b^+)^q}{(\log b^+)^{q+\beta}} \leq \frac{\tau^q}{(\log \tau)^{q+\beta}} \left(\frac{\tau^+}{\tau} \right)^q = a_\tau \lambda_\tau \left(\frac{\tau^+}{\tau} \right)^q, \quad (3.39)$$

where we used that $b^+ \leq \tau^+$ for the inequality. We can also bound the difference of the potential values at sites $Z_\tau^{(1)}$ and $Z_\tau^{(2)}$ by using Lemma 3.2.9(i). Namely, we know that if τ is large enough, for some $\beta' = 1 + \frac{1}{\alpha-d} + \frac{\varepsilon}{2}$

$$\xi(Z_\tau^{(1)}) - \xi(Z_\tau^{(2)}) \geq \frac{\tau}{\tau^+ - \tau} a_\tau (\log \tau)^{-\beta'}.$$

Hence, substituting this estimate into (3.38) together with the previous estimate (3.39) yields

$$\begin{aligned} \frac{b^+ - b^-}{\tau} &\leq \frac{1}{2 \xi(Z_\tau^{(1)}) - \xi(Z_\tau^{(2)})} \left(a_{b^+} \lambda_{b^+} \frac{\tau^+}{\tau} + a_\tau \lambda_\tau \right) \\ &\leq \frac{\tau^+ - \tau}{\tau} (\log \tau)^{\beta' - \beta} \left(\left(\frac{\tau^+}{\tau} \right)^{q+1} + 1 \right) \leq 2 \left(\frac{\tau^+}{\tau} \right)^{q+2} (\log \tau)^{\beta' - \beta}. \end{aligned}$$

It remains to bound the term τ^+/τ . By Proposition 3.2.7, we can choose $\delta = \frac{1}{d} + \frac{\varepsilon}{2(q+2)}$ such that

$$\frac{\tau_+}{\tau} = 1 + \frac{\tau^+ - \tau}{\tau} \leq (\log \tau)^\delta.$$

Finally, we have shown that if b^- is large enough

$$\frac{b^+ - b^-}{\tau} \leq 2 \left(\frac{\tau^+}{\tau} \right)^{q+2} (\log \tau)^{\beta' - \beta} \leq 2 (\log \tau)^{\beta' - \beta + (q+2)\delta} < (\log \tau)^{-\varepsilon},$$

which completes the proof. \square

Finally, we are in the position to translate the results from Section 3.2.2 from the setting of the variational problem to prove Theorem 1.2.3 for the residual lifetime function of the maximizer of the solution u .

Proof of Theorem 1.2.3. Suppose $t \mapsto h(t)$ is a nondecreasing function such that

$$\int_1^\infty \frac{dt}{th(t)^d} < \infty.$$

Without loss of generality, we can assume that there exists $\gamma' > 0$ such that $h(t) \leq (\log t)^{\gamma'}$ for all $t > 0$. Also, let $\gamma > 1 + \frac{2}{\alpha-d}$. Fix $\varepsilon > 0$ and choose $\beta > 1 + \frac{q+2}{d} + \frac{1}{\alpha-d}$ large enough such that

$$\delta := \frac{1}{4} \left(\beta - \left(1 + \frac{q+2}{d} + \frac{1}{\alpha-d} \right) \right) > \gamma' + \gamma.$$

Next, we define $\mathcal{E} := \mathcal{E}(\beta)$ as in (3.29) and denote by $[b_n^-, b_n^+]$, $n \geq 1$, the connected

components of \mathcal{E} . By Lemmas 3.2.8 and 3.2.10, we know that each of the processes $(X_t^{(1)})_{t \geq t_0}$ and $(Z_t^{(1)})_{t \geq t_0}$ jumps only at times in \mathcal{E} and each interval $[b_n^-, b_n^+]$ contains exactly one jump time, which we denote by σ_n for $X_t^{(1)}$ and τ_n for $Z_t^{(1)}$. Since we are only interested in the asymptotics of the sequences of $(\sigma_n)_{n \geq 1}$ and $(\tau_n)_{n \geq 1}$, there is no loss of generality by coupling the indices in this way. By Lemma 3.2.9 and Proposition 3.2.7, we know that for all n sufficiently large

$$2(\log \tau_n)^{-\gamma} \leq \frac{\tau_{n+1} - \tau_n}{\tau_n} \leq \frac{\varepsilon}{3} h(\frac{1}{2}\tau_n) \leq \frac{1}{2}(\log \tau_n)^{\gamma'}. \quad (3.40)$$

We now want to translate the upper bound to the jump times (σ_n) . For this purpose, we can invoke Lemma 3.2.11 to find that by our choice of β and δ we have that for all n sufficiently large

$$\frac{b_n^+ - b_n^-}{\tau_n} \leq (\log \tau_n)^{-\delta}. \quad (3.41)$$

Now, we first use that $|\sigma_n - \tau_n| \leq b_n^+ - b_n^-$ and then the estimates (3.40) and (3.41) to obtain

$$\begin{aligned} \frac{R(\sigma_n)}{\sigma_n h(\sigma_n)} &= \frac{\sigma_{n+1} - \sigma_n}{\sigma_n h(\sigma_n)} \leq \frac{\tau_{n+1} - \tau_n + b_{n+1}^+ - b_{n+1}^- + b_n^+ - b_n^-}{\tau_n(1 - (\log \tau_n)^{-\delta})h(\tau_n(1 - (\log \tau_n)^{-\delta}))} \\ &\leq \left(\frac{\tau_{n+1} - \tau_n}{\tau_n} + \frac{b_{n+1}^+ - b_{n+1}^-}{\tau_{n+1}} \frac{\tau_{n+1}}{\tau_n} + \frac{b_n^+ - b_n^-}{\tau_n} \right) \\ &\quad \times ((1 - (\log \tau_n)^{-\delta})h(\tau_n(1 - (\log \tau_n)^{-\delta})))^{-1} \\ &\leq \left(\frac{\tau_{n+1} - \tau_n}{\tau_n} + (\log \tau_{n+1})^{-\delta+\gamma'} + (\log \tau_n)^{-\delta} \right) \left(\frac{1}{2} h(\frac{1}{2}\tau_n) \right)^{-1} \\ &\leq 3 \frac{\tau_{n+1} - \tau_n}{h(\frac{1}{2}\tau_n)\tau_n} \leq \varepsilon, \end{aligned}$$

eventually for all n . In particular, this shows that, almost surely,

$$\limsup_{n \rightarrow \infty} \frac{R(\sigma_n)}{\sigma_n h(\sigma_n)} = 0.$$

However, since R jumps only at the points σ_n and decreases on $[\sigma_n, \sigma_{n+1})$, this immediately implies the first part of Theorem 1.2.3, see also Figure 1-5.

For the *second part* of the proof, suppose $t \mapsto h(t)$ is a non-decreasing function such that

$$\int_1^\infty \frac{dt}{th(t)^d} = \infty.$$

Fix $\kappa > 0$, then by Proposition 3.2.7, we know that there exists a sequence t_n such that eventually for all n

$$\frac{R^V(t_n)}{t_n h(2t_n)} \geq 3\kappa,$$

Define a subsequence of the jump times (τ_n) by choosing n_k such that for some index j we have that $t_j \in [\tau_{n_k}, \tau_{n_k+1})$. In particular, since R^V is decreasing on the interval

$[\tau_{n_k}, \tau_{n_k+1})$, we can deduce that for k large enough

$$\frac{\tau_{n_k+1} - \tau_{n_k}}{\tau_{n_k} h(2\tau_{n_k})} = \frac{R^V(\tau_{n_k})}{\tau_{n_k} h(2\tau_{n_k})} \geq \frac{R^V(t_j)}{t_j h(2t_j)} \geq 3\kappa,$$

Similarly as for the upper bound, we can estimate

$$\begin{aligned} \frac{R(\sigma_{n_k})}{\sigma_{n_k} h(\sigma_{n_k})} &= \frac{\sigma_{n_k+1} - \sigma_{n_k}}{\sigma_{n_k} h(\sigma_{n_k})} \geq \frac{\tau_{n_k+1} - \tau_{n_k} - (b_{n_k+1}^+ - b_{n_k+1}^-) - (b_{n_k}^+ - b_{n_k}^-)}{(\tau_{n_k} + (b_{n_k}^+ - b_{n_k}^-))h(\tau_{n_k} + b_{n_k}^+ - b_{n_k}^-)} \\ &\geq \left(1 - \frac{b_{n_k+1}^+ - b_{n_k+1}^-}{\tau_{n_k+1}} \frac{\tau_{n_k+1}}{\tau_{n_k}} \frac{\tau_{n_k}}{\tau_{n_k+1} - \tau_{n_k}} - \frac{b_{n_k}^+ - b_{n_k}^-}{\tau_{n_k}} \frac{\tau_{n_k}}{\tau_{n_k+1} - \tau_{n_k}}\right) \\ &\quad \times \frac{\tau_{n_k+1} - \tau_{n_k}}{\tau_{n_k}} \left((1 + (\log \tau_{n_k})^{-\delta}) h(\tau_{n_k} (1 + (\log \tau_{n_k})^{-\delta})) \right)^{-1} \\ &\geq \frac{\tau_{n_k+1} - \tau_{n_k}}{\tau_{n_k}} (1 - (\log \tau_{n_k+1})^{\gamma+\gamma'-\delta} - (\log \tau_{n_k})^{\gamma-\delta}) (2h(2\tau_{n_k}))^{-1} \\ &\geq \frac{1}{3} \frac{\tau_{n_k+1} - \tau_{n_k}}{\tau_{n_k} h(2\tau_{n_k})} \geq \kappa, \end{aligned}$$

eventually for all k large enough. This implies that

$$\limsup_{t \rightarrow \infty} \frac{R(t)}{th(t)} \geq \kappa,$$

thus completing the proof of Theorem 1.2.3. \square

3.3 A functional scaling limit theorem

The aim of this section is to prove Theorem 1.2.6 about the functional scaling limit of the maximizer of the solution u . As in previous sections, we will start by dealing with the maximizer of the variational problem. More precisely, we will prove a limit theorem for the process

$$\left(\left(\frac{Z_{tT}}{r_T}, \frac{\Phi_{tT}}{a_T} \right) : t > 0 \right)$$

by first showing convergence of the finite-dimensional distributions in Section 3.3.1 and tightness in Section 3.3.2. Then, in the final Section 3.3.3, we will use that

$$\frac{\xi(z)}{a_T} = \frac{\Phi_T(z)}{a_T} + \frac{d}{\alpha - d} \frac{|z|}{r_T} + \text{error},$$

to transfer the results to the maximizer of the profile and the potential value at that site to show Theorem 1.2.6.

First of all, we will define a suitable topology following the presentation in [Res08]. Note that by definition the limiting process $(Y_t)_{t>0}$ is an element of the space $D(0, \infty) := D((0, \infty), \mathbb{R}^{d+1})$ of all càdlàg processes defined on $(0, \infty)$ taking values in \mathbb{R}^{d+1} . We need to equip this space with the right topology. For fixed $0 < \varepsilon < M$, we can define the

standard Skorokhod metric on the space $D[\varepsilon, M] := D([\varepsilon, M], \mathbb{R}^{d+1})$ of càdlàg functions defined on $[\varepsilon, M]$ and taking values in \mathbb{R}^{d+1} , by setting for $x, y \in D([\varepsilon, M])$,

$$\text{dist}_{[\varepsilon, M]}(x, y) = \inf_{\lambda \in \Lambda_{[\varepsilon, M]}} \left(\sup_{t \in [\varepsilon, M]} |\lambda(t) - t| \vee \left(\sup_{t \in [\varepsilon, M]} |x(t) - y(\lambda(t))| \right) \right),$$

where $\Lambda_{[\varepsilon, M]}$ denotes the set of all permissible time changes, i.e. all functions $\lambda : [\varepsilon, M] \rightarrow [\varepsilon, M]$ that are continuous and strictly increasing. It is important to note that there is a equivalent metric which turns $D([\varepsilon, M])$ into a complete metric space, for more details see [Bil99]. There is a natural extension for this metric for functions defined on $(0, \infty)$. Denote for $x \in D(0, \infty)$ by $r_{[\varepsilon, M]}x$ its restriction to the interval $[\varepsilon, M]$. Then, we can define for $x, y \in D(0, \infty)$

$$\text{dist}(x, y) = \int_0^1 ds \int_{t=1}^\infty e^{-t} (\text{dist}_{[s, t]}(r_{[s, t]}x, r_{[s, t]}y) \wedge 1) dt.$$

This convergence has the feature that $x_n \rightarrow x$ in $D(0, \infty)$ if and only if $r_{[\varepsilon, M]}x_n \rightarrow r_{[\varepsilon, M]}x$ for all $0 < \varepsilon < M$. In particular, there is also an analogue criterion for weak convergence in $D(0, \infty)$. Indeed, let $\{X_n, n \geq 0\}$, X be random elements of $D(0, \infty)$ and write

$$\mathcal{T}_X = \{t > 0 : \text{Prob}\{X(t-) = X(t)\} = 1\}.$$

Then $X_n \Rightarrow X$ in $D(0, \infty)$ if and only if for all $0 < \varepsilon < M$ with $\varepsilon, M \in \mathcal{T}_X$, we have $r_{[\varepsilon, M]}X_n \Rightarrow r_{[\varepsilon, M]}X$. Note that for our limiting process Y , we find that $\mathcal{T}_Y = (0, \infty)$, since it does not have any fixed discontinuities.

By the construction of the topology we can concentrate on processes restricted to the interval $[\varepsilon, M]$. So the main part of this section is devoted to the proof of the following theorem stated in terms of the maximizer of the variational problem.

Theorem 3.3.1. *For any $0 < \varepsilon < M$, we find that as $T \rightarrow \infty$*

$$\left(\left(\frac{Z_{tT}}{r_T}, \frac{\Phi_{tT}(Z_{tT})}{a_T} \right) : t \in [\varepsilon, M] \right) \Rightarrow \left((Y_t^{(1)}, Y_t^{(2)} + q(1 - \frac{1}{t})|Y_t^{(1)}|) : t \in [\varepsilon, M] \right),$$

in the sense of weak convergence on $D([\varepsilon, M], \mathbb{R}^{d+1})$.

3.3.1 Finite-dimensional distributions

The next lemma shows that the finite-dimensional distributions of this process converge weakly to those of the limiting process defined in terms of $Y = (Y^{(1)}, Y^{(2)})$. Note since for any fixed t the set $\{Y(t) \neq Y(t-)\}$ has Prob-measure 0, the set \mathcal{T}_Y , whose complement contains the fixed discontinuities of Y , is equal to the interval $(0, \infty)$. Thus, we need to show convergence of the finite-dimensional distribution for any $0 < t_1 < t_2 < \dots < t_k < \infty$.

Lemma 3.3.2. Fix $0 < t_1 < \dots < t_k < \infty$. Then as $T \rightarrow \infty$,

$$\begin{aligned} & \left(\left(\frac{Z_{t_1 T}}{r_T}, \frac{\Phi_{t_1 T}(Z_{t_1 T})}{a_T} \right), \dots, \left(\frac{Z_{t_k T}}{r_T}, \frac{\Phi_{t_k T}(Z_{t_k T})}{a_T} \right) \right) \\ & \Rightarrow ((Y_{t_1}^{(1)}, Y_{t_1}^{(2)} + q(1 - \frac{1}{t_1})|Y_{t_1}^{(1)}|), \dots, (Y_{t_k}^{(1)}, Y_{t_k}^{(2)} + q(1 - \frac{1}{t_k})|Y_{t_k}^{(1)}|)). \end{aligned}$$

Proof. First notice, by the continuous mapping theorem, see e.g. [Bil99, Thm. 2.7], we can equivalently show that for $Y_t = (Y_t^{(1)}, Y_t^{(2)})$

$$\begin{aligned} & \left(\left(\frac{Z_{t_1 T}}{r_T}, \frac{\Phi_{t_1 T}(Z_{t_1 T})}{a_T} - q(1 - \frac{1}{t_1}) \frac{|Z_{t_1 T}|}{r_T} \right), \dots, \left(\frac{Z_{t_k T}}{r_T}, \frac{\Phi_{t_k T}(Z_{t_k T})}{a_T} - q(1 - \frac{1}{t_k}) \frac{|Z_{t_k T}|}{r_T} \right) \right) \\ & \Rightarrow (Y_{t_1}, \dots, Y_{t_k}). \end{aligned}$$

Define

$$H^* = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : y > -q(1 - \frac{1}{t_k})|x|\}.$$

Now, we know $\Phi_{tT}(Z_{tT}) > 0$, so so that together with the observation that $(Y_t^{(1)}, Y_t^{(2)}) \in H^*$ for all $t \in [t_1, t_k]$, we can conclude that it suffices to show that for any $A \subset (H^*)^k$ with $\text{Leb}_{k(d+1)}(\partial A) = 0$, we have that as $T \rightarrow \infty$

$$\text{Prob}\left\{\left(\frac{Z_{t_i T}}{r_T}, \frac{\Phi_{t_i T}(Z_{t_i T})}{a_T} - q(1 - \frac{1}{t_i}) \frac{|Z_{t_i T}|}{r_T}\right)_{i=1}^k \in A\right\} \rightarrow \text{Prob}\{(Y_{t_i})_{i=1}^k \in A\}. \quad (3.42)$$

The remainder of the proof is organised as follows: first, we show that in fact it suffices to show (3.42) for A intersected with large boxes. Secondly, we also show that it is enough to consider the maximizer of the variational problem on a large region. These steps let us express the probability in question in terms of the point process $\Pi_T = \{(z/r_t, \Phi_T(z)/a_T) : z \in \mathbb{Z}^d\}$ restricted to a relatively compact set, so that we can invoke the weak convergence of $\Pi_T \Rightarrow \Pi$ and recognize the resulting event in terms of the process Y_t .

Step 1. Define a large region

$$B_N = \{(x, y) \in H^* : |x| \leq N, \frac{1}{N} - q|x| \leq y \leq N\}.$$

We claim that we can restrict ourselves to looking at this region, in the sense that we only have to show that

$$\text{Prob}\left\{\left(\frac{Z_{t_i T}}{r_T}, \frac{\Phi_{t_i T}(Z_{t_i T})}{a_T} - q(1 - \frac{1}{t_i}) \frac{|Z_{t_i T}|}{r_T}\right)_{i=1}^k \in B_N^k \cap A\right\} \rightarrow \text{Prob}\{(Y_{t_i})_{i=1}^k \in B_N^k \cap A\}, \quad (3.43)$$

for all N in order to deduce (3.42). Indeed, we can argue similarly as in Lemma 3.1.2,

$$\begin{aligned} & \text{Prob}\left\{\left(\frac{Z_{t_i T}}{r_T}, \frac{\Phi_{t_i T}(Z_{t_i T})}{a_T} - q(1 - \frac{1}{t_i}) \frac{|Z_{t_i T}|}{r_T}\right)_{i=1}^k \in A\right\} \\ & = \text{Prob}\left\{\left(\frac{Z_{t_i T}}{r_T}, \frac{\Phi_{t_i T}(Z_{t_i T})}{a_T} - q(1 - \frac{1}{t_i}) \frac{|Z_{t_i T}|}{r_T}\right)_{i=1}^k \in B_N^k \cap A\right\} \\ & \quad + \text{Prob}\left\{\left(\frac{Z_{t_i T}}{r_T}, \frac{\Phi_{t_i T}(Z_{t_i T})}{a_T} - q(1 - \frac{1}{t_i}) \frac{|Z_{t_i T}|}{r_T}\right)_{i=1}^k \in A \setminus B_N^k\right\}. \end{aligned}$$

But, using that by [KLMS09, Lemma 3.2] $\Phi_t(Z_t)$ is an increasing function of t for all t large enough., we can bound the latter summand by

$$\begin{aligned}
& \text{Prob}\left\{\text{for some } i : \left(\frac{Z_{t_i T}}{r_T}, \frac{\Phi_{t_i T}(Z_{t_i T})}{a_T} - q\left(1 - \frac{1}{t_i}\right)\frac{|Z_{t_i T}|}{r_T}\right) \notin B_N\right\} \\
& \leq \sum_{i=1}^k \text{Prob}\left\{\frac{|Z_{t_i T}|}{r_T} > N\right\} + \text{Prob}\left\{\frac{\Phi_{t_i T}(Z_{t_i T})}{a_T} < \frac{1}{N}\right\} + \text{Prob}\left\{\frac{\Phi_{t_i T}(Z_{t_i T})}{a_T} + \frac{q}{t}\frac{|Z_{t_i T}|}{r_T} > N\right\} \\
& \leq k \left[\max_{i=1, \dots, k} \text{Prob}\left\{\frac{|Z_{t_i T}|}{r_{t_i T}} > N \frac{r_T}{r_{t_i T}}\right\} + \text{Prob}\left\{\frac{\Phi_{t_1 T}(Z_{t_1 T})}{a_{t_1 T}} < \frac{1}{N} \frac{a_T}{a_{t_1 T}}\right\} \right. \\
& \quad \left. + \text{Prob}\left\{\frac{\Phi_{t_k T}(Z_{t_k T})}{a_{t_k T}} > \frac{1}{2} N \frac{a_T}{a_{t_k T}}\right\} + \max_{i=1, \dots, k} \text{Prob}\left\{\frac{|Z_{t_i T}|}{a_{t_i T}} > \frac{N t_1}{2q} \frac{r_T}{r_{t_k T}}\right\} \right] \\
& \leq C_1 k \left[\left(\frac{N}{t_k^{q+1}}\right)^{d-\alpha} + e^{-C_2(N t_1^q)^{\alpha-d}} + \left(\frac{N}{2t_k^q}\right)^{d-\alpha} + \left(\frac{N t_1}{2q t_k^{q+1}}\right)^{d-\alpha} \right],
\end{aligned} \tag{3.44}$$

where $C_1, C_2 > 0$ are some constants and in the last step we used Lemma 3.2.3 and the fact that $\frac{a_T}{a_{t_1 T}} \rightarrow t_1^{-q}$ and $\frac{r_T}{r_{t_k T}} \rightarrow t_k^{-(q+1)}$. Hence, the error bounds in the last display tend to 0 uniformly in t as we let $N \rightarrow \infty$. Similarly, if we look at the probabilities for Y , we see that

$$\text{Prob}\{(Y_{t_i})_{i=1}^k \in A\} = \text{Prob}\{(Y_{t_i})_{i=1}^k \in A \cap B_N^k\} + \text{Prob}\{(Y_{t_i})_{i=1}^k \in A \setminus B_N^k\}.$$

Again, we can bound the latter probability from above by

$$\begin{aligned}
& \text{Prob}\{(Y_{t_i})_{i=1}^k \in A \setminus B_N^k\} \leq \sum_{i=1}^k \text{Prob}\{Y_{t_i} \notin B_N\} \\
& \leq \sum_{i=1}^k \left[\text{Prob}\{|Y_{t_i}^{(1)}| > N\} + \text{Prob}\{-q\left(1 - \frac{1}{t_k}\right)|Y_{t_i}^{(1)}| \leq Y_{t_i}^{(2)} \leq \frac{1}{N} - q|Y_{t_i}^{(1)}|\} \right. \\
& \quad \left. + \text{Prob}\{Y_{t_i}^{(2)} > N\} \right] \\
& \leq k \left[\text{Prob}\{|Y_{t_k}^{(1)}| > N\} + \text{Prob}\{|Y_{t_1}^{(1)}| \leq \frac{t_k}{qN}\} + \text{Prob}\{Y_1^{(2)} > N\} \right],
\end{aligned}$$

where we used that $|Y_t^{(1)}|$ is an increasing function in t and $Y_1^{(2)} \geq Y_t^{(2)}$ for all t , since Y_1 corresponds to the point of the Poisson point process Π with the largest second component. Finally, by the construction of Y based on Π it is clear that all the probabilities tend to 0 as $N \rightarrow \infty$. Therefore, we can conclude that if we can show (3.43) we can also deduce (3.42).

Step 2. Denote, for $K > N$ by $Z_{tT}^{K,T}$ the point satisfying

$$\Phi_{tT}(Z_{tT}^{K,T}) = \max\{\Phi_{tT}(z) : z \text{ such that } t\xi(z) \geq z \text{ and } \left(\frac{z}{r_T}, \frac{\Phi_T(z)}{a_T}\right) \in B_K\},$$

where in case of a tie we take the one with the larger ℓ^1 norm. We claim that if K is large, $Z_{tT}^{K,T}$ agrees with high probability with the global maximizer Z_{tT} . Indeed, we

find that

$$\begin{aligned}
& \left| \text{Prob}\left\{ \left(\frac{Z_{t_i T}}{r_T}, \frac{\Phi_{t_i T}(Z_{t_i T})}{a_T} - q\left(1 - \frac{1}{t_i}\right) \frac{|Z_{t_i T}|}{r_T} \right)_{i=1}^k \in B_N^k \cap A \right\} \right. \\
& \quad \left. - \text{Prob}\left\{ \left(\frac{Z_{t_i T}^{K,T}}{r_T}, \frac{\Phi_{t_i T}(Z_{t_i T}^{K,T})}{a_T} - q\left(1 - \frac{1}{t_i}\right) \frac{|Z_{t_i T}^{K,T}|}{r_T} \right)_{i=1}^k \in B_N^k \cap A \right\} \right| \\
& \leq \text{Prob}\{\text{there exists } i : Z_{t_i T}^{K,T} \neq Z_{t_i T}\} \leq \sum_{i=1}^k \text{Prob}\left\{ \left(\frac{Z_{t_i T}}{r_T}, \frac{\Phi_{t_i T}(Z_{t_i T})}{a_T} \right) \notin B_K \right\} \quad (3.45) \\
& \leq k \max_{i=1, \dots, k} \left[\text{Prob}\left\{ \frac{|Z_{t_i T}|}{r_T} \geq K \right\} + \text{Prob}\left\{ \frac{\Phi_T(Z_T)}{r_T} > K \right\} + \text{Prob}\left\{ \frac{\Phi_{t_i T}(Z_{t_i T})}{a_T} < \frac{1}{K} \right\} \right],
\end{aligned}$$

where for the last term, we use that by Lemma 3.1.3, we can express

$$\frac{\Phi_{t_i T}(Z_{t_i T})}{a_T} = \frac{\Phi_T(Z_{t_i T})}{a_T} + q\left(1 - \frac{1}{t_i}\right) \frac{|Z_{t_i T}|}{r_T} + \text{error}(T),$$

where the error term tends to 0. Hence, as in (3.44), we can use Lemma 3.2.3 to show that the expression (3.45) tends to zero if we first let T and then $K \rightarrow \infty$.

Step 3. Using the point process we want to express the probability

$$\begin{aligned}
& \text{Prob}\left\{ \left(\frac{Z_{t_i T}^{K,T}}{r_T}, \frac{\Phi_{t_i T}(Z_{t_i T}^{K,T})}{a_T} - q\left(1 - \frac{1}{t_i}\right) \frac{|Z_{t_i T}^{K,T}|}{r_T} \right)_{i=1}^k \in B_N^k \cap A \right\} \\
& = \int_{A \cap B_N^k} \text{Prob}\left\{ \frac{Z_{t_i T}^{K,T}}{r_T} \in dx_i, \frac{\Phi_{t_i T}(Z_{t_i T}^{K,T})}{a_T} - q\left(1 - \frac{1}{t_i}\right) \frac{|Z_{t_i T}^{K,T}|}{r_T} \in dy_i \text{ for all } i \right\}, \quad (3.46)
\end{aligned}$$

in the limit as $T \rightarrow \infty$. First note that by Lemma 3.1.3 we have that for any $t \in [t_1, t_k]$

$$\frac{\Phi_{tT}(z)}{a_T} = \frac{\Phi_T(z)}{a_T} + q\left(1 - \frac{1}{t}\right) \frac{|z|}{r_T} + \delta_{1-t}\left(T, \frac{z}{r_T}, \frac{\Phi_T(z)}{a_T}\right),$$

where the error δ_{1-t} goes to 0 uniformly for all z such that $(\frac{z}{r_T}, \frac{\Phi_T(z)}{a_T}) \in B_K$ and also uniformly for all $t \in [t_1, t_k]$. Recall also that $\Pi_T = \sum_{x \in \mathbb{Z}^d: t\xi(z) > |z|} \delta_{(x/r_T, \Phi_T(x)/a_T)}$ converges weakly to Π on H^* . Now, as the restriction to large boxes ensures that we are only dealing with the point process on relatively compact sets, we can in the limit as $T \rightarrow \infty$ express the condition

$$\frac{Z_{t_i T}^{K,T}}{r_T} = x_i, \frac{\Phi_{t_i T}(Z_{t_i T}^{K,T})}{a_T} - q\left(1 - \frac{1}{t_i}\right) \frac{|Z_{t_i T}^{K,T}|}{r_T} = y_i$$

by requiring that Π has an atom in (x_i, y_i) and all other points (x, y) of Π restricted to B_K satisfy

$$y + q\left(1 - \frac{1}{t_i}\right)|x| \leq y_i + q\left(1 - \frac{1}{t_i}\right)|x_i|.$$

Therefore, if we denote by $\mathcal{C}_{t_i}(x_i, y_i)$ the open cone of all points $(x, y) \in H^*$ satisfying

$$y + q\left(1 - \frac{1}{t_i}\right)|x| > y_i + q\left(1 - \frac{1}{t_i}\right)|x_i|,$$

we can express the probability in (3.46) in the limit as

$$\begin{aligned} & \lim_{T \rightarrow \infty} \text{Prob} \left\{ \left(\frac{Z_{t_i T}^{K,T}}{r_T}, \frac{\Phi_{t_i T}(Z_{t_i T}^{K,T})}{a_T} - q(1 - \frac{1}{t_i}) \frac{|Z_{t_i T}^{K,T}|}{r_T} \right)_{i=1}^k \in B_N^k \cap A \right\} \\ &= \int_{A \cap B_N^k} \text{Prob} \left\{ \Pi|_{B_K}(\mathrm{d}x_i \mathrm{d}y_i) = 1 \text{ for } i = 1, \dots, k, \Pi|_{B_K} \left(\bigcup_{i=1}^k \mathcal{C}_{t_i}(x_i, y_i) \right) = 0 \right\}. \end{aligned}$$

Now, we can remove the restriction of the point process to B_K , by letting $K \rightarrow \infty$ and noting that the probability that for some $(x_i, y_i) \in A \cap B_N^k$ and some $i = 1, \dots, k$ the point process Π has a point in the set $\mathcal{C}_{t_i}(x_i, y_i) \cap B_K^c$ can be bounded from above by the probability that Π has a point in the set

$$\{(x, y) \in \mathbb{R}^{d+1} : y > \frac{1}{N} - q(1 - \frac{1}{t_k})|x| \text{ and } (y > K \text{ or } |x| > K)\}.$$

But the intensity measure ν of Π gives finite mass to this region, so that we can conclude that the probability of the latter event tends to zero as $K \rightarrow \infty$. Hence, we can combine this observation with the estimate in (3.45) and letting first $T \rightarrow \infty$ and then $K \rightarrow \infty$, to deduce that

$$\begin{aligned} & \lim_{T \rightarrow \infty} \text{Prob} \left\{ \left(\frac{Z_{t_i T}}{r_T}, \frac{\Phi_{t_i T}(Z_{t_i T})}{a_T} - q(1 - \frac{1}{t_i}) \frac{|Z_{t_i T}|}{r_T} \right)_{i=1}^k \in B_N^k \cap A \right\} \\ &= \int_{A \cap B_N^k} \text{Prob} \left\{ \Pi(\mathrm{d}x_i \mathrm{d}y_i) = 1 \text{ for } i = 1, \dots, k, \Pi \left(\bigcup_{i=1}^k \mathcal{C}_{t_i}(x_i, y_i) \right) = 0 \right\} \\ &= \text{Prob} \{ (Y_{t_i})_{i=1}^k \in B_N^k \cap A \}, \end{aligned}$$

where in the last step we used the definition of Y . For an illustration of the event under the integral, see also Figure 3-2. Thus, together with the first step, we have completed the proof of the lemma. \square

3.3.2 Tightness

Now, fix $0 < \varepsilon < M$. Before we start with the tightness argument, we have to prove two auxiliary lemmas. The first says that if we rescale time linearly, we can control the probability that the maximizer makes any small jumps on any compact interval. The second lemma uses this result to show that during the interval $[\varepsilon T, MT]$ the ℓ^1 norm of the maximizer is sufficiently well approximated by the ℓ^1 norm of Z_{MT} .

Lemma 3.3.3. *Let τ_i denote the jump times of the process $(Z_t)_{t \geq \varepsilon T}$ in increasing order. Then*

$$\liminf_{T \rightarrow \infty} \text{Prob} \{ \text{for all jump times } \tau_i \in [\varepsilon T, MT] : \tau_{i+1} - \tau_i \geq \delta T \} \geq p(\delta),$$

where $p(\delta) \rightarrow 1$ as $\delta \downarrow 0$.

point process on \widehat{H} with intensity

$$\nu(dx dy) = \frac{\alpha dx dy}{(y + q|x|)^{\alpha+1}}.$$

Then, in the limit as $T \rightarrow \infty$ we know that if we fix $(\frac{Z_T}{r_T}, \frac{\Phi_T(Z_T)}{a_T}) = (x, y)$ then the probability that $(Z_t : t \geq T)$ jumps more than once in the interval $[T, (1 + \delta)T]$ is bounded from above by the probability that the point process Π has no points in the set $D_0(|x|, y)$ and at least two points in the set $D_\delta(|x|, y) \setminus D_0(|x|, y)$. To make this bound absolutely rigorous, one has restrict the process $(Z_t/r_t, \Phi_t(Z_t)/a_t)$ to large boxes, let $T \rightarrow \infty$ and then the size of the boxes go to infinity and finally justify interchanging the limit. This is a very similar calculation to Lemma 3.1.2 and Lemma 3.3.2 and is therefore omitted. Using this observation, we obtain the bound

$$\begin{aligned} & \limsup_{T \rightarrow \infty} \text{Prob}\{Z_t \text{ jumps more than once in the interval } [T, (1 + \delta)T]\} \\ & \leq \int_{y \geq 0} \int_{x \in \mathbb{R}^d} \text{Prob}\{\Pi(dx dy) = 1, \Pi(D_0(|x|, y)) = 0, \Pi(D_\delta(|x|, y) \setminus D_0(|x|, y)) \geq 2\} \\ & = \int_{y \geq 0} \int_{x \in \mathbb{R}^d} e^{-\nu(D_0(|x|, y))} (1 - e^{-f_\delta(x, y)} - f_\delta(x, y) e^{-f_\delta(x, y)}) \nu(dx dy), \end{aligned}$$

where $f_\delta(x, y) = \nu(D_\delta(|x|, y)) - \nu(D_0(|x|, y))$. It remains to be shown that this expression divided by δ converges to 0. As we would like to invoke the dominated convergence theorem, we want to show that the integrand is bounded by an integrable function. Thus we divide by δ and use that $1 - e^{-x} \leq x$, for $x \geq 0$, to obtain

$$\begin{aligned} & \int_{y \geq 0} \int_{x \in \mathbb{R}^d} e^{-\nu(D_0(|x|, y))} \frac{1}{\delta} (1 - e^{-f_\delta(x, y)} - f_\delta(x, y) e^{-f_\delta(x, y)}) \nu(dx dy) \\ & \leq \int_{y \geq 0} \int_{x \in \mathbb{R}^d} e^{-\nu(D_0(|x|, y))} \frac{1}{\delta} f_\delta(x, y) (1 - e^{-f_\delta(x, y)}) \nu(dx dy). \end{aligned} \tag{3.47}$$

We can estimate $1 - e^{-f_\delta(x, y)} \leq 1$ uniformly in x, y , and then continue as in the proof of Proposition 3.1.4, by first substituting $r = x_1 + \dots + x_d$ and $u_i = x_i$ for $i = 2, \dots, d$ and in the next step $y + qr = \frac{y}{v}$ to find an upper bound on the previous display

$$\begin{aligned} & \int_{y \geq 0} \int_{x \in \mathbb{R}^d} e^{-\nu(D_0(|x|, y))} \frac{1}{\delta} (\nu(D_\delta(|x|, y)) - \nu(D_0(|x|, y))) \nu(dx dy) \\ & = \frac{2^d}{(d-1)!} \int_{y \geq 0} \int_{r \geq 0} e^{-\nu(D_0(r, y))} \frac{1}{\delta} (\nu(D_\delta(r, y)) - \nu(D_0(r, y))) \frac{\alpha r^{d-1} dr}{(y + qr)^{\alpha+1}} dy \\ & = \frac{2^d \alpha \vartheta}{q^d (d-1)!} \int_{y \geq 0} \int_{v \in (0, 1)} v^{\alpha-d} (1-v)^{d-1} y^{2(d-\alpha)-1} \frac{1}{\delta} (\varphi_\delta(v)^{-1} - \varphi_0(v)^{-1}) e^{-\vartheta y^{d-\alpha}} dv dy, \end{aligned} \tag{3.48}$$

where in the last step we used the expression $\nu(D_\delta(r, y)) = \vartheta y^{d-\alpha} \varphi_\delta(v)^{-1}$ from (3.9) with φ_δ given by (3.7) and $\vartheta = \frac{2^d B(\alpha-d, d)}{q^d (d-1)!}$. Next, we estimate the part of the integrand

that depends on δ , so for $\tilde{B}(x) := \tilde{B}(x, \alpha - d, d)$ we consider

$$\begin{aligned} \frac{1}{\delta}(\varphi_\delta(v)^{-1} - \varphi_0(v)^{-1}) &= \frac{1}{\delta} \left(\frac{(1+\delta)^\alpha}{(\delta+v)^{\alpha-d}} v^{\alpha-d} \tilde{B}\left(\frac{v+\delta}{1+\delta}\right) - \tilde{B}(v) \right) \\ &\leq \frac{1}{\delta}((1+\delta)^\alpha - 1) + \frac{1}{\delta}(\tilde{B}\left(\frac{v+\delta}{1+\delta}\right) - \tilde{B}(v)). \end{aligned}$$

As the first term is $\leq 2\alpha$ for all $\delta \leq \delta_0$ for some small δ_0 (independent of x, y), we can concentrate on the second term. Now, we can use the definition of \tilde{B} to write

$$\begin{aligned} \frac{1}{\delta}(\tilde{B}\left(\frac{v+\delta}{1+\delta}\right) - \tilde{B}(v)) &= \frac{1}{\delta} \int_v^{\frac{v+\delta}{1+\delta}} u^{\alpha-d-1} (1-u)^{d-1} du \leq \frac{1}{\delta} \int_v^{\frac{v+\delta}{1+\delta}} u^{\alpha-d-1} du \\ &\leq \frac{1}{\delta} \left(\frac{v+\delta}{1+\delta} - v \right) \max\{v^{\alpha-d-1}, 1\} \leq \max\{v^{\alpha-d-1}, 1\}. \end{aligned}$$

Then, we can substitute this expression back to get an upper bound on (3.48) of the form

$$\begin{aligned} \frac{2^d \alpha \vartheta}{q^d (d-1)!} \int_{y \geq 0} \int_{v \in (0,1)} v^{\alpha-d} (1-v)^{d-1} y^{2(d-\alpha)-1} 3\alpha \max\{v^{\alpha-d-1}, 1\} e^{-\vartheta y^{d-\alpha}} dv dy \\ = \frac{3 \cdot 2^d \alpha^2 \vartheta}{q^d (d-1)!} \int_0^1 v^{\alpha-d} (1-v)^{d-1} \max\{v^{\alpha-d-1}, 1\} dv \int_0^\infty y^{2(d-\alpha)-1} e^{-\vartheta y^{d-\alpha}} dy \\ \leq \frac{3 \cdot 2^d \alpha^2}{q^d (d-1)! (\alpha-d)} \max\{B(\alpha-d+1, d), B(2(\alpha-d), d)\}. \end{aligned}$$

This shows in particular, that we can invoke the dominated convergence theorem and in the expression (3.47) we can take the limit as $\delta \rightarrow 0$ under the integral and use that $\frac{1}{\delta} f_\delta(x, y)$ and $1 - e^{-f_\delta(x, y)}$ tend to 0 pointwise to obtain that

$$\begin{aligned} \lim_{\delta \downarrow 0} \limsup_{T \rightarrow \infty} \text{Prob}\{Z_t \text{ jumps more than once in the interval } [T, (1+\delta)T]\} \\ \leq \int_{y \geq 0} \int_{x \in \mathbb{R}^d} e^{-\nu(D_0(|x|, y))} \lim_{\delta \downarrow 0} \left(\frac{1}{\delta} f_\delta(x, y) (1 - e^{-f_\delta(x, y)}) \right) \nu(dx dy) = 0, \end{aligned}$$

which completes the proof of the lemma. \square

The previous lemma gives us some control about the maximum number of jumps that Z_t can make during an interval $[\varepsilon T, MT]$. We will use this to show that the probability that the ℓ^1 -norm of the rescaled version Z_t can never be too large.

Lemma 3.3.4. *For fixed ε, M , we have that*

$$\lim_{\kappa \rightarrow \infty} \limsup_{T \rightarrow \infty} \text{Prob}\left\{ \sup_{t \in [\varepsilon T, MT]} \frac{|Z_t|}{r_T} \geq \kappa \right\} = 0.$$

Proof. We start by considering what happens at a jump time τ of Z_t . In that case, we know by Lemma 3.1.8 that $\xi(Z_\tau^{(1)}) > \xi(Z_\tau^{(2)})$. In particular, we have, using that $\chi(z) = x - \rho \log x$ is increasing on $x > \rho$,

$$\Phi_\tau(Z_\tau^{(1)}) \geq \xi(Z_\tau^{(1)}) - \frac{1}{\tau} |Z_\tau^{(1)}| \log \xi(Z_\tau^{(1)}) > \xi(Z_\tau^{(2)}) - \frac{1}{\tau} |Z_\tau^{(1)}| \log \xi(Z_\tau^{(2)}).$$

Since $\Phi_\tau(Z_\tau^{(1)}) = \Phi_\tau(Z_\tau^{(2)})$, we thus obtain that

$$\xi(Z_\tau^{(2)}) - \frac{|Z_\tau^{(2)}|}{\tau} \log \xi(Z_\tau^{(2)}) + \frac{1}{\tau} \eta(Z_\tau^{(2)}) > \xi(Z_\tau^{(2)}) - \frac{1}{\tau} |Z_\tau^{(1)}| \log \xi(Z_\tau^{(2)}).$$

Hence using that $\eta(z) \leq |z| \log d$, we find that

$$|Z_\tau^{(2)}| < |Z_\tau^{(1)}| \left(1 - \frac{\log d}{\log \xi(Z_\tau^{(2)})}\right)^{-1} < |Z_\tau^{(1)}| \left(1 - \frac{\log d}{q \log \tau(1+o(1))}\right)^{-1},$$

where we invoked [KLMS09, Lemma 3.2] to deduce that eventually for all t , $\xi(Z_t^{(2)}) > a_t(\log t)^{-1}$. Hence, if we denote by N_T the number of jumps of Z_t in the interval $[\varepsilon T, MT]$, we have that for T large enough

$$\sup_{t \in [\varepsilon T, MT]} |Z_t| \leq (1 - \frac{2 \log d}{q \log \varepsilon T})^{-N_T} |Z_{MT}|. \quad (3.49)$$

Fix $\varepsilon' > 0$. Then, by Lemma 3.3.3, we know that we can choose $\delta > 0$ such that if (τ_i) denote the jump times of Z_t in $[\varepsilon T, MT]$,

$$\liminf_{T \rightarrow \infty} \text{Prob}\{\text{for all jump times } \tau_i \in [\varepsilon T, MT] : \tau_{i+1} - \tau_i \geq \delta T\} \geq 1 - \frac{\varepsilon'}{4}.$$

But on the event that all jump times τ_i in $[\varepsilon T, MT]$ satisfy $\tau_{i+1} - \tau_i \geq \delta T$, we know that $N_T \leq \frac{M-\varepsilon}{\delta} + 1$. Therefore, we can deduce that on this event,

$$\sup_{t \in [\varepsilon T, MT]} |Z_t| \leq (1 - \frac{2 \log d}{q \log \varepsilon T})^{-\frac{M-\varepsilon}{\delta}-1} |Z_{MT}|.$$

Hence, for any $\kappa > 1$, we can estimate that

$$\begin{aligned} & \text{Prob}\left\{\sup_{t \in [\varepsilon T, MT]} \frac{|Z_t|}{r_T} \geq \kappa\right\} \\ & \leq \text{Prob}\left\{(1 - \frac{2 \log d}{q \log \varepsilon T})^{-\frac{M-\varepsilon}{\delta}-1} \frac{|Z_{MT}|}{r_T} \geq \kappa\right\} + \text{Prob}\{\text{for some } \tau_i : \tau_{i+1} - \tau_i < \delta T\} \\ & \leq \text{Prob}\left\{\frac{|Z_{MT}|}{r_{MT}} \geq \kappa M^{-(q+1)}(1 + o(1))\right\} + \frac{\varepsilon'}{2} \\ & \leq (1 + \frac{\varepsilon'}{2}) \text{Prob}\{|Y_1^{(1)}| \geq \kappa M^{-(q+1)}\} + \frac{\varepsilon'}{2}, \end{aligned}$$

for all t sufficiently large, where we use that $Z_t/r_t \Rightarrow Y_1^{(1)}$. Hence, by choosing κ large enough the latter expression can be made smaller than ε' , which completes the proof. \square

To prove tightness, we will use the following characterization (taken e.g. from [Bil99, Thm. 13.2]). Let P_n be a sequence of probability measures on $D[a, b]$. The sequence $\{P_n\}$ is tight if and only if the following two conditions are satisfied

$$\begin{aligned} & \text{(i)} \quad \lim_{\kappa \rightarrow \infty} \limsup_{n \rightarrow \infty} P_n\{x : \|x\| \geq \kappa\} = 0 \\ & \text{(ii)} \quad \text{for each } \varepsilon' > 0 : \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} P_n\{x : w'_x(\delta) \geq \varepsilon'\} = 0. \end{aligned} \quad (3.50)$$

Here, $\|x\|$ is the uniform norm, i.e.

$$\|x\| = \sup_{t \in [a, b]} |x(t)|,$$

and the modulus $w'_x(\delta)$ is defined as

$$w'_x(\delta) = \inf_{\{t_i\}} \max_{1 \leq i \leq v} w_x[t_{i-1}, t_i],$$

where the infimum runs over all partitions $a = t_0 < t_1 < \dots < t_v = 1$ of $[a, b]$ satisfying $\min_{1 \leq i \leq v} (t_i - t_{i-1}) > \delta$ and w_x is the modulus of continuity defined for an interval $I \subset [a, b]$ as

$$w_x(I) = \sup_{s, t \in I} |x(s) - x(t)|.$$

Lemma 3.3.5. *The family of probability measures $\{\text{Prob}_T\}$ is tight, where Prob_T is the law of*

$$V_T = \left(\left(\frac{Z_{tT}}{r_T}, \frac{\Phi_{tT}(Z_{tT})}{a_T} \right) : t \in [\varepsilon T, MT] \right),$$

under Prob .

Proof. We have to check the two conditions in (3.50).

(i) First recall from [KLMS09, Lemma 3.2] that eventually for all t , the function $t \mapsto \Phi_t(Z_t)$ is increasing, so that we can assume throughout the proof that this property holds for all $t \geq \varepsilon T$. Note that

$$\|V_T\| = \sup_{t \in [\varepsilon, M]} \left\{ \left| \frac{Z_{tT}}{r_T} \right| + \left| \frac{\Phi_{tT}(Z_{tT})}{a_T} \right| \right\} = \sup_{t \in [\varepsilon, M]} \left\{ \frac{|Z_{tT}|}{r_T} \right\} + \frac{\Phi_{MT}(Z_{MT})}{a_T}.$$

Therefore, we find that for any $\kappa > 0$

$$\text{Prob}\{\|V_T\| \geq \kappa\} \leq \text{Prob}\left\{ \sup_{t \in [\varepsilon, M]} \frac{|Z_{tT}|}{r_T} \geq \frac{\kappa}{2} \right\} + \text{Prob}\left\{ \frac{\Phi_{MT}(Z_{MT})}{a_T} \geq \frac{\kappa}{2} \right\}. \quad (3.51)$$

Now, by Lemma 3.3.4 and the weak convergence of $\Phi_t(Z_t)/a_t \Rightarrow Y_1^{(2)}$, we can deduce that the above expressions tend to zero, if we first let $T \rightarrow \infty$ and then $\kappa \rightarrow \infty$.

(ii) Fix $\delta > 0$ and a partition $(t_i)_{i=0}^v$ of $[\varepsilon, M]$ such that $\delta < t_{i+1} - t_i < 2\delta$ and such that all the jump times of $(Z_{tT} : t \in [\varepsilon, M])$ are some of the t_i . This is possible if all the jump times τ_i of Z_t in $[\varepsilon T, MT]$ satisfy $\tau_{i+1} - \tau_i \geq \delta T$, an event which by the previous Lemma 3.3.3 has probability tending to 1 if we first let $T \rightarrow \infty$ and then $\delta \rightarrow 0$. Thus, we can work on this event from now on.

First, using that Z_{tT} does not jump in $[t_{i-1}, t_i)$ and the fact that $\Phi_t(Z_t)$ is increasing

and $t \mapsto \xi(Z_t)$ non-decreasing by Lemma 3.1.8, we can estimate

$$\begin{aligned}
w_{V_T}[t_{i-1}, t_i] &= \sup_{s, t \in [t_{i-1}, t_i]} \left| \frac{Z_{tT}}{r_T} - \frac{Z_{sT}}{r_T} \right| + \sup_{t, s \in [t_{i-1}, t_i]} \left| \frac{\Phi_{tT}(Z_{tT})}{a_T} - \frac{\Phi_{sT}(Z_{sT})}{a_T} \right| \\
&= \frac{1}{a_T} (\Phi_{t_i T}(Z_{t_{i-1} T}) - \Phi_{t_{i-1} T}(Z_{t_{i-1} T})) \\
&= \frac{1}{a_T} \left(\frac{1}{t_{i-1} T} - \frac{1}{t_i T} \right) (|Z_{t_{i-1}}| \log \xi(Z_{t_{i-1}}) - \eta(Z_{t_{i-1}})) \\
&\leq \frac{2\delta}{\varepsilon^2} \sup_{s \in [\varepsilon, M]} \left\{ \frac{|Z_{sT}|}{r_T} \right\} \frac{\log \xi(Z_{MT})}{\log T}.
\end{aligned}$$

Now, recall that, by (3.32), we can bound $\xi(Z_t) \leq a_t \log t$ eventually for all t so that together with $\log a_T = (q + o(1)) \log T$ we obtain

$$w'_{V_T}(\delta) \leq \frac{2\delta}{\varepsilon^2} \sup_{s \in [\varepsilon, M]} \left\{ \frac{|Z_{sT}|}{r_T} \right\} \frac{\log \xi(Z_{MT})}{\log T} \leq \frac{2q\delta}{\varepsilon^2} \sup_{s \in [\varepsilon, M]} \left\{ \frac{|Z_{sT}|}{r_T} \right\} (1 + o(1)).$$

Finally, we can use Lemma 3.3.4 to deduce that

$$\lim_{\delta \downarrow 0} \limsup_{T \rightarrow \infty} \text{Prob}\{w'_{V_T}(\delta) \geq \epsilon'\} \leq \lim_{\delta \downarrow 0} \limsup_{T \rightarrow \infty} \text{Prob}\left\{ \frac{2q\delta}{\varepsilon^2} \sup_{s \in [\varepsilon, M]} \left\{ \frac{|Z_{sT}|}{r_T} \right\} (1 + o(1)) \geq \epsilon' \right\} = 0,$$

so that also the second part of the criterion (3.50) is satisfied. \square

3.3.3 Functional limit theorem for the maximizer of the solution profile

In this section, we will prove Theorem 1.2.6 by translating the functional limit theorem from the maximizer of the variational problem to the maximizer of the solution u . We will prove both parts (a) and (b) simultaneously. Therefore, we will extend our topology to the space of càdlàg functions $f : (0, \infty) \rightarrow \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$. The main argument is contained in the next lemma.

Lemma 3.3.6. *As $T \rightarrow \infty$, the Skorokhod distance*

$$\text{dist}_{[\varepsilon, M]} \left(\left(\frac{Z_{tT}}{r_T}, \frac{\Phi_t(Z_{tT})}{a_T}, \frac{\Phi_{tT}(Z_{tT})}{a_T} + \frac{q}{t} \frac{|Z_{tT}|}{r_T} \right)_{t \in [\varepsilon, M]}, \left(\frac{X_{tT}}{r_T}, \frac{1}{a_T} \frac{\log U(tT)}{tT}, \frac{\xi(X_{tT})}{a_T} \right)_{t \in [\varepsilon, M]} \right),$$

tends to 0 in probability.

Proof. The first step is to set up a *time change* that relates X_t and Z_t . Recall from the discussion in Section 3.2.3 that if t_0 is large enough then the jump times of $(X_t)_{t \geq t_0}$ and $(Z_t)_{t \geq t_0}$ always occur in pairs σ, τ which are close together, in the sense that we can choose $\beta > 1 + \frac{1}{\alpha-d}$ and then each connected component of the set $\mathcal{E}(\beta)$, defined in (3.29), contains exactly one jump time of each of the two processes. In particular by Lemma 3.2.11 there exists $\delta > 0$ such that

$$\frac{|\sigma - \tau|}{\tau} \leq (\log \tau)^{-\delta}. \quad (3.52)$$

From now on suppose $T \geq t_0/\varepsilon$. Let $0 < \gamma < \frac{M-\varepsilon}{4}$ and note that by Proposition 3.1.1, we have that the event $Z_{\varepsilon T} = Z_{\varepsilon T(1+\gamma)}$ and $Z_{MT(1-\gamma)} = Z_{MT}$ has probability tending to 1 if we first let $T \rightarrow \infty$ and then $\gamma \downarrow 0$. Hence, we can work on this event.

Then, let $(\sigma_i, i = 0, \dots, N)$ be the jump times of X_t and $(\tau_i, i = 0, \dots, N)$ the jump times of Z_t that lie in the interval $[\varepsilon T, MT]$ with the convention that σ_i and τ_i are paired in the above sense. In particular, we can assume that all the jump times satisfy

$$\frac{|\sigma_i - \tau_i|}{\tau_i} \leq (\log \tau_i)^{-\delta} \leq (\log \varepsilon T)^{-\delta} < \gamma,$$

so that if we assume $Z_{\varepsilon T} = Z_{\varepsilon T(1+\gamma)}$ and $Z_{MT(1-\gamma)} = Z_{MT}$, we know that all pairs $\sigma_i, \tau_i \in (\varepsilon T, MT)$. Now, we can set up a time change that relates X_t and Z_t as follows. Let $s_i = \sigma_i/T$ and $t_i = \tau_i/T$. Define $\lambda : [\varepsilon, M] \rightarrow \mathbb{R}$ such that $\lambda(\varepsilon) = \varepsilon$, $\lambda(M) = M$ and such that $\lambda(s_i) = t_i$ for all $i = 0, \dots, N$ and on each interval $[s_{i-1}, s_i]$ we require that λ is linear. First, we can estimate the deviation of the time-change from the identity by

$$\begin{aligned} \sup_{t \in [\varepsilon, M]} |\lambda t - t| &= \sup_{i=0, \dots, N} |\lambda(s_i) - s_i| = \sup_{i=0, \dots, N} \frac{1}{T} |\tau_i - \sigma_i| \leq M \sup_{i=0, \dots, N} \frac{|\tau_i - \sigma_i|}{\tau_i} \\ &\leq M \sup_{i=0, \dots, N} (\log \tau_i)^{-\delta} \leq M (\log \varepsilon T)^{-\delta}, \end{aligned} \quad (3.53)$$

which converges to 0 as $T \rightarrow \infty$. This shows that the time-change is sufficiently close to the identity.

Since, we have equipped our target space with the ℓ^1 -norm, we can consider each component of the process individually. For the *first component*, we notice that the time-change is set up in such way that $X_{tT} = Z_{\lambda(t)T}$, which shows that

$$\sup_{t \in [\varepsilon, M]} \left| \frac{Z_{\lambda(t)T}}{r_T} - \frac{X_{tT}}{r_T} \right| = 0.$$

For the *second component*, we need to consider

$$\begin{aligned} &\frac{1}{a_T} \left| \frac{\log U(tT)}{tT} - \Phi_{\lambda(t)T}(Z_{\lambda(t)T}) \right| \\ &\leq \frac{1}{a_T} \left| \frac{\log U(tT)}{tT} - \Phi_{tT}(Z_{tT}) \right| + \frac{1}{a_T} \left| \Phi_{tT}(Z_{tT}) - \Phi_{\lambda(t)T}(Z_{tT}) \right| \\ &\quad + \frac{1}{a_T} \left| \Phi_{\lambda(t)T}(Z_{tT}) - \Phi_{\lambda(t)T}(Z_{\lambda(t)T}) \right|, \end{aligned} \quad (3.54)$$

and show that each of the expressions tends to 0 as $T \rightarrow \infty$ (uniformly for all $t \in [\varepsilon, M]$).

For the first term, we can use Propositions 4.2 and 4.4 from [KLMS09] to conclude that there exists $\delta' > 0$ and $C > 0$ such that almost surely eventually for all t

$$\Phi_t(Z_t) - 2d + o(1) \leq \frac{1}{t} \log U(t) \leq \Phi_t(Z_t) + Ct^{q-\delta'}.$$

In particular, this shows that the first term in (3.54) tends to 0 (uniformly for all $t \in [\varepsilon, M]$). For the second term, we can use for the first inequality the bound $\eta(z) \leq |z| \log d$, for the second the above bound for the time-change (3.53) and also that we

can find a $\delta' > 0$ such that $a_t(\log t)^{-\delta'} \leq \xi(Z_t) \leq a_t(\log t)^{\delta'}$, which follows from (3.32) combined with [KLMS09, Lemma 3.2]. Hence, we can write for T large enough and all $t \in [\varepsilon, M]$,

$$\begin{aligned} \frac{1}{a_T} |\Phi_{tT}(Z_{tT}) - \Phi_{\lambda(t)T}(Z_{tT})| &= \frac{1}{a_T} \left| \frac{1}{tT} - \frac{1}{\lambda(t)T} \right| |Z_{tT}| \log \xi(Z_{tT}) - \eta(Z_{tT})| \\ &\leq \frac{1}{\varepsilon^2} |\lambda(t) - t| \sup_{t \in [\varepsilon, M]} \frac{|Z_{tT}|}{r_T \log T} \max\{|\log \xi(Z_{tT})|, 2d\} \\ &\leq (1 + o(1)) \frac{M}{\varepsilon^2} (\log \varepsilon T)^{-\delta} \sup_{t \in [\varepsilon, M]} \frac{|Z_{tT}|}{r_T \log T}, \end{aligned}$$

so that the latter expression tends to 0 in probability by Lemma 3.3.4.

Now, in order to deal with the last term in (3.54), note that if $t \in (s_i \vee t_i, s_{i+1} \wedge t_{i+1})$ for some $i = 0, \dots, N-1$, then $Z_{tT} = Z_{\lambda(t)T}$ so that the term vanishes. Otherwise if $t \in [s_i \wedge t_i, s_i \vee t_i]$, then tT is in the set of transition times \mathcal{E} as discussed in Section 3.2.3 and we find that $\{Z_{tT}, Z_{\lambda(t)T}\} \subset \{Z_{\lambda(t)T}^{(1)}, Z_{\lambda(t)T}^{(2)}\}$ and also that there exists $\beta > 1 + \frac{1}{\alpha-d}$ such that

$$\begin{aligned} \frac{1}{a_T} |\Phi_{\lambda(t)T}(Z_{tT}) - \Phi_{\lambda(t)T}(Z_{\lambda(t)T})| &\leq \frac{1}{a_T} (\Phi_{\lambda(t)T}(Z_{\lambda(t)T}^{(1)}) - \Phi_{\lambda(t)T}(Z_{\lambda(t)T}^{(2)})) \\ &\leq \frac{a_{\lambda(t)T}}{a_T} (\log \lambda(t)T)^{-\beta} \\ &\leq M^q (1 + o(1)) (\log \varepsilon T)^{-\beta}, \end{aligned}$$

so that we can conclude that in probability the expression in (3.54) for the second component tends to 0 uniformly for all $t \in [\varepsilon, M]$.

Finally, we have to consider the *third component* of the process. In this case, we first recall that by (3.32) and by [KLMS09, Lemma 3.2], there exists $\delta' > 0$ such that eventually for all t , $a_t(\log t)^{-\delta'} \leq \xi(Z_t) \leq a_t(\log t)^{\delta'}$. Hence, using that $Z_{tT} = X_{\lambda^{-1}(t)T}$, we can estimate

$$\begin{aligned} \left| \frac{\Phi_{tT}(Z_{tT})}{a_T} + \frac{q}{t} \frac{|Z_{tT}|}{r_T} - \frac{\xi(X_{\lambda^{-1}(t)T})}{a_T} \right| &= \left| \frac{q}{t} \frac{|Z_{tT}|}{r_T} - \frac{|Z_{tT}|}{t r_T} \frac{\log \xi(Z_{tT})}{\log T} + \frac{\eta(Z_{tT})}{t r_T \log T} \right| \\ &\leq \sup_{t \in [\varepsilon, M]} \frac{|Z_{tT}|}{r_T} \frac{1}{\varepsilon} C' \frac{\log \log MT}{\log T}, \end{aligned}$$

where C' is some constant (depending only on ε, M). Again, by Lemma 3.3.4, in probability the latter expression tends to 0 uniformly in $t \in [\varepsilon, M]$, which completes the proof of the lemma. \square

Proof of Theorem 1.2.6. By a classic result on weak convergence, see e.g. [Bil99, Thm. 3.1], the previous lemma ensures that the process

$$\left(\left(\frac{X_{tT}}{r_T}, \frac{1}{a_T} \frac{\log U(tT)}{tT}, \frac{\xi(X_{tT})}{a_T} \right) : t \in [\varepsilon, M] \right),$$

has the same weak limit as

$$\left(\left(\frac{Z_{tT}}{r_T}, \frac{\Phi_t(Z_{tT})}{a_T}, \frac{\Phi_{tT}(Z_{tT})}{a_T} + \frac{q}{t} \frac{|Z_{tT}|}{r_T} \right) : t \in [\varepsilon, M] \right),$$

which was identified in Theorem 3.3.1, as

$$\left((Y_t^{(1)}, Y_t^{(2)} + \frac{d}{\alpha-d}(1 - \frac{1}{t})|Y_t^{(1)}|, Y_t^{(2)} + \frac{d}{\alpha-d}|Y_t^{(1)}|) : t \in [\varepsilon, M] \right).$$

Hence, projecting onto the first and third, respectively second, component completes the proof. \square

Proof of Proposition 1.2.4. We concentrate on the one-dimensional distribution, since the higher dimensional case works analogously. Let f be a continuous, bounded, non-negative function on \mathbb{R}^d , in particular there exists $\kappa > 0$ such that $f(x) \leq \kappa$ for all $x \in \mathbb{R}^d$. In order to show that

$$\xi_{tT} := \sum_{x \in \mathbb{Z}^d} v(tT, x) \delta_{\frac{x}{r_T}} \Rightarrow \delta_{Y_t},$$

it suffices to show that the Laplace functionals converge, i.e. as $T \rightarrow \infty$

$$\mathbb{E}[e^{-\xi_{tT}(f)}] = \mathbb{E}[e^{-\sum_x v(tT, x) f(\frac{x}{r_T})}] \rightarrow \mathbb{E}[e^{-f(Y_t)}].$$

Let $\varepsilon > 0$ and for $\delta = \min\{\frac{\log(1+\frac{\varepsilon}{2})}{\kappa}, \frac{\varepsilon}{4}\}$, consider the event

$$A_\delta = \{v(tT, Z_{tT}) > 1 - \delta\}.$$

Then, since $v(Z_{tT}) \Rightarrow 1$, we can choose T_0 large enough such that for all $T \geq T_0$, $\text{Prob}(A_\delta) > 1 - \frac{\varepsilon}{4}$. Therefore, we can estimate

$$\begin{aligned} \mathbb{E}[e^{-f(Y_t)}] - \mathbb{E}[e^{-\xi_{tT}}] &\leq \mathbb{E}[e^{-f(Y_t)}] - \mathbb{E}[e^{-\sum_{x \in \mathbb{Z}^d} v(tT, x) f(\frac{x}{r_T})} \mathbb{1}_{A_\delta}] \\ &\leq \mathbb{E}[e^{-f(Y_t)}] - \mathbb{E}[e^{-f(\frac{Z_{tT}}{r_T})}] e^{-\delta} \\ &\leq \mathbb{E}[e^{-f(Y_t)}] - \mathbb{E}[e^{-f(\frac{Z_{tT}}{r_T})}] + \frac{\varepsilon}{2}. \end{aligned}$$

Similarly, for a lower bound,

$$\begin{aligned} \mathbb{E}[e^{-f(Y_t)}] - \mathbb{E}[e^{-\xi_{tT}(f)}] &\geq \mathbb{E}[e^{-f(Y_t)}] - \mathbb{E}[e^{-\sum_{x \in \mathbb{Z}^d} v(tT, x) f(\frac{x}{r_T})} \mathbb{1}_{A_\delta}] - \frac{\varepsilon}{4} \\ &\geq \mathbb{E}[e^{-f(Y_t)}] - \mathbb{E}[e^{-(1-\delta)f(\frac{Z_{tT}}{r_T})}] - \frac{\varepsilon}{4} \\ &\geq \mathbb{E}[e^{-f(Y_t)}] - \mathbb{E}[e^{-f(\frac{Z_{tT}}{r_T})}] - \frac{\varepsilon}{2}. \end{aligned}$$

Hence, we can invoke Theorem 3.3.1, the weak convergence of $\frac{Z_{tT}}{r_T}$ to $Y_t := Y_t^{(1)}$, to complete the proof. \square

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